REGULARITY OF CONSERVATIVE INDUCTIVE LIMITS

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(Received 31 July 1998)

ABSTRACT. A sequentially complete inductive limit of Fréchet spaces is regular, see [3]. With a minor modification, this property can be extended to inductive limits of arbitrary locally convex spaces under an additional assumption of conservativeness.

Keywords and phrases. Regular and conservative inductive limits of locally convex spaces.

1991 Mathematics Subject Classification. Primary 46A13; Secondary 46A30.

Throughout the paper \( E_1 \subset E_2 \subset \cdots \) is a sequence of Hausdorff locally convex spaces with continuous identity maps \( \text{id}: E_n \rightarrow E_{n+1}, \ n \in \mathbb{N} \). Their respective topologies are denoted by \( \tau_n \). The topology of their inductive limit \( \text{ind} E_n \) is denoted by \( \tau = \text{ind} \tau_n \).

We will use a result from [1, Cor. IV.6.5]. It reads:

If \( F \) as well as all spaces \( E_n \) are Fréchet and \( T: F \rightarrow \text{ind} E_n \) is a linear map with a closed graph, then there is \( n \in \mathbb{N} \) such that \( T \) is a continuous map of \( F \) into \( E_n \).

According to [2, Sec. 5.2], the space \( \text{ind} E_n \) is called \( \alpha \)-regular, resp. regular, if every set bounded in \( \text{ind} E_n \) is contained, resp. bounded, in some constituent space \( E_n \). We will need a slightly modified notion of regularity.

**Definition 1.** An inductive limit \( \text{ind} E_n \) is quasi \( \alpha \)-regular, resp. quasi regular, if every set bounded in \( \text{ind} E_n \) is a subset of a \( \tau \)-closure of a set contained, resp. bounded, in some constituent space \( E_n \).

**Definition 2.** An inductive limit \( \text{ind} E_n \) is called conservative if for every linear subspace \( F \subset \text{ind} E_n \), we have

\[
\text{ind} (F \cap E_n, \tau_n) = (F, \text{ind} \tau_n).
\]

**Lemma.** Let a locally convex (Hausdorff) space \( E \) be sequentially complete, and \( B \) be a balanced, bounded, closed, and convex set in \( E \). Then the linear span \( F \) of \( B \), equipped with the topology generated by the Minkowski functional of \( B \), is a Banach space and the identity map \( \text{id}: F \rightarrow E \) is continuous.

**Proof.** Clearly \( F \) is a normed space and \( \text{id}: F \rightarrow E \) is continuous.

To prove the completeness of \( F \), take a Cauchy sequence \( \{x_n\} \) in \( F \). Since \( \text{id}: F \rightarrow E \) is continuous, \( \{x_n\} \) is Cauchy in \( E \). Hence it converges to some \( x \in E \). The set \( \bigcup \{x_n; n \in \mathbb{N}\} \), which is bounded in \( F \), is contained in some \( \alpha B \). Since the set \( \alpha B \) is closed in \( E \), we have \( x \in \alpha B \subset F \).

For any \( 0 \)-nbhd \( \lambda B, \lambda > 0, \) in \( F \), there exists \( k \in \mathbb{N} \) such that \( m, n > k \) imply \( x_n - x_m \in \lambda B \). If we let \( m \rightarrow \infty \), we get \( x_n - x \in \lambda B \) for \( n > k \), i.e., \( x_n \rightarrow x \) in \( F \).
**Proposition 1.** Any sequentially complete ind \( E_n \) is quasi \( \alpha \)-regular.

**Proof.** Let a set \( A \) be bounded in ind \( E_n \). Denote by \( B \) its balanced, convex, \( \tau \)-closed hull, and by \( F \) the linear span of \( B \) with the same topology \( \gamma \) as in the Lemma. We know that \( F \) is a Banach space.

For any \( n \in \mathbb{N} \), denote by \( G_n \) the completion of the normed space \( (F \cap E_n, \gamma) \). Then \( G_n \subset F \) and \( F \) equals strict inductive limit ind \( G_n \). Since \( B \) is bounded in \( F \), it is bounded in ind \( G_n \). Hence, by [1, Cor. IV. 6.5], \( B \) is bounded in some \( G_n \).

Finally, \( A \subset B \) and \( B \) is a \( \gamma \)-closure of a set \( V = \bigcup \{E_n \cap \lambda B; 0 < \lambda < 1 \} \) in \( F \cap E_n \). Hence \( A \) is also a subset of the \( \tau \)-closure of \( V \) in ind \( E_n \).

**Proposition 2.** Let ind \( E_n \) be sequentially complete and conservative. Then every set \( A \subset E_1 \), which is bounded in ind \( E_n \) is also bounded in some constituent space \( E_n \).

**Proof.** Take such \( A \) and assume that it is not bounded in any \( E_n \). Then for any \( n \in \mathbb{N} \), there exists a balanced convex 0-nbhbd \( U_n \) in \( E_n \) which does not absorb \( A \). For any \( m, n \in \mathbb{N} \), choose \( a_{m,n} \in A \) such that \( a_{m,n} \notin mU_n \). Denote by \( B \) the \( \tau \)-closure of the convex balanced hull of \( \bigcup \{a_{m,n}; m, n \in \mathbb{N} \} \) and by \( F \) the linear span of \( B \). For any \( m, n \in \mathbb{N} \), there exists \( f_{m,n} \in (\text{ind } E_n)' \), (the dual of ind \( E_n \)), such that \( f_{m,n}(a_{m,n}) \neq 0 \). Put \( V_{m,n} = \{x \in F; |f_{m,n}(x)| \leq 1 \} \) and denote by \( F_n \) the linear space \( F \) equipped with the topology generated by \( \{U_m; m \geq n\} \cup \{V_{m,n}; m, n \in \mathbb{N} \} \). Then each \( F_n \) is a metrizable Hausdorff locally convex space and its completion \( G_n \) is a Fréchet space.

Finally, let \( H \) be the space \( F \) equipped with the topology generated by the Minkowski functional of \( B \). The set \( B \) is bounded in ind \( E_n \), hence, by the Lemma, \( H \) is Banach space and the identity map id : \( H \to \text{ind } E_n \) is continuous.

Since ind \( E_n \) is conservative and \( F \subset \text{ind } E_n \), we have

\[
\text{ind}(F, \tau_n) = (F, \text{ind } \tau_n) \quad (2)
\]

For any \( n \in \mathbb{N} \), the identity maps \( (F, \tau_n) \to F_n \to G_n \) are continuous. Hence

\[
\text{id} : \text{ind}(F, \tau_n) \to \text{ind } G_n \quad (3)
\]

is continuous, too. Then, the continuity of \( \text{id} : H \to \text{ind } E_n \) implies the continuity of \( \text{id} : H \to (F, \text{ind } \tau_n) \). By (2) and (3), we finally get the continuity of \( \text{id} : H \to \text{ind } G_n \).

By [1, Cor. IV. 6.5], there exists \( n \in \mathbb{N} \) such that \( \text{id} : H \to G_n \) is continuous. Since the set \( B \) is bounded in \( H \) and contained in \( F_n \), it is bounded in \( G_n \), and also bounded in \( F_n \). But then \( B \), as well as its subset \( A \), are absorbed by the 0-nbhd \( V_n \) in \( F_n \), a contradiction.

By combining Propositions 1 and 2, we get

**Theorem.** Any sequentially complete conservative ind \( E_n \) is quasi regular.

**Corollary.** If moreover each space \( E_n \) in the above Theorem is closed in ind \( E_n \), then ind \( E_n \) is regular.

**References**


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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

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