ALMOST TRIANGULAR MATRICES OVER
DEDEKIND DOMAINS

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Abstract. Every matrix over a Dedekind domain is equivalent to a direct sum of matrices
\( A = (a_{i,j}) \), where \( a_{i,j} = 0 \) whenever \( j > i + 1 \).

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1. Introduction. Two \( m \times n \) matrices \( A \) and \( B \) over a ring \( R \) are called equivalent
if \( B = PAQ \) for invertible matrices \( P \) and \( Q \) over \( R \). From now on, assume that \( R \)
denotes a Dedekind domain with quotient field \( K \). If \( I = (a,b) \) is a non principal ideal
in \( R \), then, in contrast with the situation for Principal Ideal Domains, the \( 1 \times 2 \) matrix
\( [a,b] \) is not equivalent over \( R \) to a matrix whose off diagonal entries are 0. Using the
separated divisor theorem in the form given by Levy in [2], other facts about matrices
over Dedekind domains in [2], and elementary properties of ideals in Dedekind domain
[1], we show that any \( m \times n \) matrix over a Dedekind domain is equivalent to a direct
sum of matrices \( A = (a_{i,j}) \) with \( a_{i,j} = 0 \) when \( j > i + 1 \). If the direct summand
\( A \) has rank \( r \), then the number of rows, respectively columns, of \( A \) is either \( r \) or \( r + 1 \).
The corresponding result for similarity of matrices over principal ideal rings is that
every \( n \times n \) matrix over a principal ideal ring is similar to an upper triangular matrix
[3, p. 42].

2. Diagonalization of matrices. If \( A \) is an \( m \times n \) matrix, then \( A \) can be viewed as
an \( R \)-module homomorphism \( A : R^n \to R^m \) by left multiplication. If \( M_A \) denotes the
submodule of \( R^m \) generated by the columns of \( A \), then \( M_A \) is the image of \( A \) in \( R^m \)
and the isomorphism class of the cokernel \( S_A = R^m / M_A \) of \( A \) determines the equivalence
class of \( A \).

Separated divisor theorem [2]. There is a chain of integral \( R \)-ideals \( L_1 \subseteq L_2 \subseteq \cdots \subseteq L_r \) and a fractional \( R \)-ideal \( H \) such that
\[
S_A = \begin{cases} \oplus_{i=1}^{r'} \frac{K}{L_i} \oplus H \oplus R^{m-r-1}, & m < r \\ \oplus_{i=1}^{r'} \frac{K}{E_i}, & m = r, \end{cases}
\]
(2.1)
where \( H = \prod_{i=1}^{r'} L_i \) if \( r = n \) and \( H \supseteq R \) if \( r = 0 \) or \( r = m \).
The isomorphism class of \( S_A \), the ideals \( \{L_i\}_{i=1}^{r'} \) (as sets), and the isomorphism class
of \( H \) both determine and are determined by the equivalence class of \( A \).
We also need the following elementary facts about ideals in Dedekind domains.

**Lemma 1** [1, p.150, 154]. Let $I,J$ be integral ideals in $R$. Then

1. There is an $\alpha$ in the quotient field $K$ of $R$ such that $\alpha I$ is integral and $\alpha I + J = R$;
2. There is an $R$-module isomorphism $\gamma : IJ \oplus R \to I \oplus J$, given by $\gamma(u,v) = (x_1 u - u, \alpha u - x_2 v)$, where $\alpha$ is as in (1) and $x_1 \in I, x_2 \in J$ are chosen with $\alpha x_1 - x_2 = 1$.

**Note.** The $R$-linear homomorphism $\gamma$ is given by the matrix $\left( \begin{array}{cc} -1 & x_1 \\ \alpha & -x_2 \end{array} \right)$, where $\alpha \in K$.

**Theorem 2.2.** Every $m \times n$ matrix $A$ over a Dedekind domain is equivalent to a direct sum of matrices $(a_{ij})$ with $a_{jj} = 0$ whenever $j > i + 1$.

**Proof.** An $m \times n$ matrix $A$ is called indecomposable if $A$ is not equivalent to a direct sum of matrices $B_1, B_2$. That is, $A$ is not equivalent to a direct sum of matrices $B_1, B_2$. If $A = 0$, the result is clear. Assume that $A \neq 0$. It is sufficient to verify the result for indecomposable matrices. In this case, if $r$ is the rank of $A$ over the quotient field $K$ of $R$, then [2, Lem. 2.1] asserts that $m = r$ or $r + 1$ and $n = r$ or $r + 1$. There are then four possible cases to check.

**Case 1.** Assume that $m = r$ and $n = r$. Then $S_A = \oplus_{i=1}^r R/L_i$, with $L_1, \ldots, L_r$ integral $R$-ideals generated by $a \in R$. Let $\phi_0 : R^r \to \oplus_{i=1}^r L_i \oplus R^{r-1}$ be given by $\phi_0(r_1, \ldots, r_r) = (ar_1, r_2, \ldots, r_r)$ and let $\phi_j : L_1 \oplus \cdots \oplus L_{j-1} \oplus \oplus_{i=j}^r L_i \oplus R \to R^{r-j+1}$ be given by $\phi_j = I_{j-1} \oplus y_j \oplus I_{r-j-1}$, where $y_j : \oplus_{i=1}^r L_i \oplus R \to L_j \oplus \oplus_{i=j+1}^r L_i$ is the map given in Lemma 1 and $I_{j-1}, I_{r-j-1}$ are the identity maps of indicated rank. Let $\phi : R^r \to L_1 \oplus \cdots \oplus L_r \subset R^r$ be given by $\phi = \phi \cap \phi_2 \cdots \cdot \phi_0$. Then the matrix $[\phi]$ may have entries which are not in $R$, $[\phi]$ has all its entries in $R$ since each $L_j$ is integral. If we write

$$[\phi_j] = \begin{pmatrix} I_j & 0 & 0 & 0 \\ 0 & -1 & x_j^1 & 0 \\ 0 & \alpha_j & -x_j^2 & 0 \\ 0 & 0 & 0 & I_{r-j-1} \end{pmatrix},$$

then a direct calculation shows that

$$[\phi] = \begin{pmatrix} -a & x_1^1 & 0 & 0 & 0 & 0 & 0 \\ -a \alpha_1 & -x_2^1 & x_1^2 & 0 & 0 & 0 & 0 \\ -a \alpha_1 \alpha_2 & \alpha_2 x_1^2 & x_2^2 & x_1^3 & 0 & 0 & 0 \\ -a \alpha_1 \alpha_2 \alpha_3 & \alpha_2 \alpha_3 x_2^2 & x_3^2 & x_1^4 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a \prod_{i=1}^{r-1} \alpha_i & \cdots & \cdots & \cdots & \alpha_{r-2} x_2^{r-2} & x_2^{r-1} \end{pmatrix}.$$  

Since $[\phi]$ has the same number of rows and columns and the same cokernel as $A$, $[\phi]$ is equivalent to $A$. 


Remark. Assume that \( L_i = \langle a_i \rangle \) is principal for each \( i, i = 1, \ldots, r \) and \( a_i \in R \). The isomorphism \( y_j : \prod_{i=j}^{r} L_i \oplus R \rightarrow L_j \oplus \prod_{i=j+1}^{r} L_i \) can be given as \( y_j(u, v) = (\alpha_j u, \beta_j v) \), where \( \alpha_j = 1 / \prod_{i=j+1}^{r} a_i \) and \( \beta_j = \prod_{i=j+1}^{r} a_i \). In this case, \( [\varphi] = \text{diag}[a_1, \ldots, a_r] \) with \( a_i | a_{i+1} \) for \( 1 \leq i \leq r \). This is the only case which occurs if \( R \) is a PID.

Case 2. Assume that \( m = r \) and \( n = r + 1 \). Then \( S_A = \oplus_{i=1}^{r} R/L_i \) with \( L_i, 1 \leq i \leq r \) integral ideals and \( L_1 \subseteq L_2 \subseteq \cdots \subseteq L_r \). Let \( L_{r+1} \) be integral ideal with \( \prod_{i=1}^{r+1} L_i = \langle a \rangle \) principal, then \( \oplus_{i=1}^{r+1} L_i \equiv R^n \) and there is a chain of \( R \)-homomorphisms

\[
R^n \xrightarrow{\varphi} L_1 \oplus \cdots \oplus L_r \oplus L_{r+1} \xrightarrow{\pi} L_1 \oplus \cdots \oplus L_r \subseteq R^r,
\]

where \( \pi \) is the projection on \( L_1 \oplus \cdots \oplus L_r \) along \( L_{r+1} \). The matrix of \( \pi \circ \varphi \) is an \( m \times n \) matrix obtained by deleting the last row of \( [\varphi] \) and, thus, has the same form as in Case 1. Since the cokernel of \( \pi \circ \varphi \) is the same as \( A \) and \( [\varphi] \) has the same number of rows and columns as \( A, [\pi \varphi] \) is equivalent to \( A \).

Case 3. Assume that \( m = r + 1 \) and \( n = r \). Then \( S_A = \oplus_{i=1}^{r} R/L_i \oplus H \), where \( L_i, 1 \leq i \leq r \) are integral ideals and \( H \equiv \prod_{i=1}^{r} L_i \). Choose \( a \in R \) with \( L_r H^{-1} a \) integral. Note that \( L_r H^{-1} a \) is a submodule of \( H^{-1} a \). From Case 1, we construct an \( R \)-isomorphism \( \varphi_r : R^r \rightarrow L_1 \oplus \cdots \oplus L_{r-1} \oplus L_r H^{-1} a \subseteq R^{r+1} \) whose matrix has the same form as that of \( [\varphi] \) in Case 1. By Lemma 1, there is a chain of isomorphisms \( \psi : H^{-1} a \oplus H \rightarrow H^{-1} Ha \oplus R \rightarrow R \) carrying \( L_r H^{-1} a \) onto a submodule \( N \) of \( R \oplus R \). By [1, Cor. 18.24], \( (H^{-1} a \oplus H)/L_r H^{-1} a \equiv R/L_r \oplus H \). Let \( \Phi = (I_{r-1} \oplus \psi) \circ \varphi_r : R^n \rightarrow R^m \). The matrix of \( \Phi \) is \( m \times n \) and the first \( r = n \) rows are the same as \( [\varphi_r] \). The last row does not contribute any entries above the main diagonal. So, for each \( j > i + 1 \), the \( i, j \)th entry of \( [\Phi] \) is 0. Since the cokernel of \( [\Phi] \) is \( S_A \) and \( [\Phi] \) has the same number of rows and columns as \( A, [\Phi] \) and \( A \) are equivalent.

Case 4. Let \( S_A = \oplus_{i=1}^{r} R/L_i \oplus H \), where \( L_1, \ldots, L_r \) are integral ideals with \( L_1 \subseteq \cdots \subseteq L_r \) and by replacing \( H \) (if necessary) by an isomorphic copy, \( H \) is an integral ideal. By [1, Thm. 18.20], there is an integral ideal \( H_0 \) with \( H_0 H \) principal and \( H_0 + H = R \). There is an \( a \in R \) such that \( J = (\prod_{i=1}^{r} L_i \cdot H_0)^{-1} a \subseteq H \). As in Case 1, there is an isomorphism \( \varphi_{r+1} : R^{r+1} \rightarrow L_1 \oplus \cdots \oplus L_{r-1} \oplus L_r H_0 \oplus J \). View \( L_1 \leq R \) for \( 1 \leq i \leq r \), \( L_i H_0 \leq H_0 \). As in Case 3, there is an isomorphism \( \psi : H_0 \oplus H \rightarrow R \) with \( \psi(L_i H_0) = N \leq R \oplus R \) and \( R \oplus R/N \equiv R/L_r \oplus H \). Let \( \Phi = (I_{r-1} \oplus \psi) \circ \varphi_{r+1} \). Then \( \Phi : R^{r+1} \rightarrow R^{r+1} \) and all the rows, except possibly the last two of \( [\Phi] \), are the same as that of \( [\varphi] \) in Case 1. So, for each \( j > i + 1 \), the \( i, j \)th entry of \( [\Phi] \) is 0. Since the cokernel of \( \Phi \) is \( S_A \), \( [\Phi] \) and \( A \) are equivalent.

Remark. While we could have given explicit formula for the entries in the matrices constructed in Cases 2, 3, and 4 as in Case 1, these entries are not canonically determined by \( A \) as a result of the many choices made in their construction. In particular, the choices of \( \alpha \) and \( x_1, x_2 \) in Lemma 1 are not canonically determined by the ideals \( I, J \).

References


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