CHARACTERIZATION ON SOME ABSOLUTE SUMMABILITY FACTORS OF INFINITE SERIES

W. T. SULAIMAN

(Received 12 June 1998)

Abstract. A general theorem concerning some absolute summability factors of infinite series is proved. This theorem characterizes as well as generalizes our previous result [4]. Other results are also deduced.

Keywords and phrases. Summability, series, sequences.

1991 Mathematics Subject Classification. 40D15, 40F15, 40F05.

1. Introduction. Let $\sum a_n$ be an infinite series with partial sum $s_n$. Let $\sigma_n^\delta$ and $\eta_n^\delta$ denote the $n$th Cesàro mean of order $\delta (\delta > -1)$ of the sequences $\{s_n\}$ and $\{a_n\}$, respectively. The series $\sum a_n$ is said to be summable $|C, \delta|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\delta - \sigma_{n-1}^\delta|^k < \infty,$$

or, equivalently,

$$\sum_{n=1}^{\infty} n^{-1} |\eta_n|^k < \infty.$$ (1.1)

Let $\{p_n\}$ be a sequence of positive real constants such that

$$P_n = \sum_{v=0}^{n} p_v \rightarrow \infty \text{ as } n \rightarrow \infty.$$ (1.3)

The series $\sum a_n$ is said to be summable $|N, p_n|_k$, $k \geq 1$, if (Bor [1])

$$\sum_{n=1}^{\infty} \left(\frac{p_n}{p_0}\right)^{k-1} |T_n - T_{n-1}|^k < \infty,$$ (1.4)

where

$$T_n = P_n^{-1} \sum_{v=0}^{n} p_v s_v.$$ (1.5)

For $p_n = 1$, $|N, p_n|_k$ summability is equivalent to $|C, 1|_k$ summability. In general, the two summabilities are not comparable. Let $\{\phi_n\}$ be any sequence of positive real constants. The series $\sum a_n$ is said to be summable $|N, p_n, \phi_n|_k$, $k \geq 1$, if (Sulaiman [4])

$$\sum_{n=1}^{\infty} \phi_n^{k-1} |T_n - T_{n-1}|^k < \infty.$$ (1.6)
Clearly,
\[
\left| \mathcal{N}, p_n, \frac{P_n}{p_n} \right|_k = |\mathcal{N}, p_n|_k, \quad |\mathcal{N}, 1, n|_k = |C, 1|_k.
\] (1.7)

**Theorem 1.1** (Sulaiman [4]). Let \{p_n\}, \{q_n\}, and \{\varphi_n\} be sequences of real positive constants. Let \( t_n \) denote the \((\mathcal{N}, p_n)\)-mean of the series \( \sum a_n \). If
\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^k \left( \frac{q_n}{Q_n} \right)^k \varphi_n^{k-1} |\epsilon_n|^k |\Delta t_{n-1}|^k < \infty,
\] (1.8)
then the series \( \sum a_n \epsilon_n \) is summable \( |\mathcal{N}, q_n, \varphi_n|_k, k \geq 1 \), where \( \Delta f_n = f_n - f_{n+1} \) for any sequence \{f_n\} and
\[
Q_n = \sum_{v=0}^{n} q_v \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty \quad (q_{-1} = Q_{-1} = 0).
\] (1.9)

2. Lemmas

**Lemma 2.1** (Bor [1]). Let \( k > 1 \) and \( A = (a_{nv}) \) be an infinite matrix. In order that \( A \in (\ell^k; \ell^k) \), it is necessary that
\[
a_{nv} = O(1) \quad (\text{all } n, v).
\] (2.1)

**Lemma 2.2.** Suppose that \( \epsilon_n = O(f_n g_n), f_n, g_n \geq 0, \{\epsilon_n / f_n g_n\} \) monotonic, \( \Delta g_n = O(1) \), and \( \Delta f_n = O(f_n / g_{n+1}) \). Then \( \Delta \epsilon_n = O(f_n) \).

**Proof.** Let \( k_n = (\epsilon_n / f_n g_n) = O(1) \). If \( (k_n) \) is nondecreasing, then
\[
\Delta \epsilon_n = k_n f_n g_n - k_{n+1} f_{n+1} g_{n+1}
\leq k_n f_n g_n - k_n f_{n+1} g_{n+1}
= k_n \Delta (f_n g_n) = k_n (f_n \Delta g_n + g_n \Delta f_n),
\] (2.2)
\[
|\Delta \epsilon_n| = O(f_n |\Delta g_n|) + O(g_{n+1} |\Delta f_n|)
= O(f_n) + O(f_n) = O(f_n).
\]

If \( (k_n) \) is nonincreasing, write \( \nabla f_n = f_{n+1} - f_n \),
\[
\nabla \epsilon_n = k_{n+1} f_{n+1} g_{n+1} - k_n f_n g_n
\leq k_n \nabla (f_n g_n)
= k_n (f_n \nabla g_n + g_{n+1} \nabla f_n),
\] (2.3)
\[
|\Delta \epsilon_n| = |\nabla \epsilon_n| = O(f_n |\nabla g_n|) + O(g_{n+1} |\nabla f_n|)
= O(f_n |\Delta g_n|) + O(g_{n+1} |\Delta f_n|)
= O(f_n) + O(f_n) = O(f_n).
\] \( \square \)
3. Main Result. We state and prove the following theorem:

**Theorem 3.1.** Let \( \{p_n\}, \{q_n\}, \{\alpha_n\}, \) and \( \{\beta_n\} \) be sequences of positive real numbers such that

\[
\left\{ \frac{\beta_n q_n}{Q_n} \right\} \text{ is nonincreasing;} \tag{3.1}
\]

\[
p_n Q_n = O \left( P_n q_n \right); \tag{3.2}
\]

\[
\left\{ \frac{P_n q_n}{p_n Q_n} \left( \frac{\beta_n}{\alpha_n} \right)^{1-(1/k)} \epsilon_n \right\} \text{ is monotonic;} \tag{3.3}
\]

\[
\Delta \left( \frac{Q_n}{q_n} \right) = O(1); \tag{3.4}
\]

\[
\Delta \left\{ \frac{p_n}{P_n} \left( \frac{\alpha_n}{\beta_n} \right)^{1-(1/k)} \right\} = O \left\{ \frac{P_n q_{n+1}}{p_n Q_{n+1}} \left( \frac{\alpha_n}{\beta_n} \right)^{1-(1/k)} \right\}. \tag{3.5}
\]

Then the necessary and sufficient conditions that \( \sum a_n \epsilon_n \) be summable \( \mid N, q_n, \beta_n \mid_k \), whenever \( \sum a_n \) is summable \( \mid N, p_n, \alpha_n \mid_k, \) \( k \geq 1 \), are

\[
\epsilon_n = O \left\{ \frac{p_n Q_n}{P_n q_n} \left( \frac{\alpha_n}{\beta_n} \right)^{1-(1/k)} \right\}, \tag{3.6}
\]

\[
\Delta \epsilon_n = \left\{ \frac{p_n}{P_{n-1}} \left( \frac{\alpha_n}{\beta_n} \right)^{1-(1/k)} \right\}. \tag{3.7}
\]

**Proof.** Write

\[
T_n = \beta_n^{1-(1/k)} \left( \frac{q_n}{Q_n Q_{n-1}} \right) \sum_{v=1}^{n} P_{v-1} a_v \epsilon_v,
\]

\[
t_n = \alpha_n^{1-(1/k)} \left( \frac{p_n}{P_n P_{n-1}} \right) \sum_{v=1}^{n} P_{v-1} a_v,
\]

\[
T_n = \beta_n^{1-(1/k)} \left( \frac{q_n}{Q_n Q_{n-1}} \right) \sum_{v=1}^{n} P_{v-1} a_v \frac{Q_{v-1}}{P_{v-1}} \epsilon_v
\]

\[
= \beta_n^{1-(1/k)} \left( \frac{q_n}{Q_n Q_{n-1}} \right) \sum_{v=1}^{n-1} \sum_{v=1}^{v} (P_{v-1} a_v) \Delta \left( \frac{Q_{v-1}}{P_{v-1}} \epsilon_v \right) + \sum_{v=1}^{n} (P_{v-1} a_v) \left( \frac{Q_{v-1}}{P_{v-1}} \epsilon_v \right)
\]

\[
= \beta_n^{1-(1/k)} \left( \frac{q_n}{Q_n Q_{n-1}} \right) \sum_{v=1}^{n-1} \sum_{v=1}^{v} (P_{v-1} a_v) \alpha_{v}^{(1/k)-1} \epsilon_v \left\{ \frac{-q_v}{P_{v-1}} \epsilon_v + \frac{P_v Q_v \epsilon_v}{P_{v-1} P_v} + \frac{Q_v \Delta \epsilon_v}{P_v} \right\}
\]

\[
+ \left( \beta_n^{1-(1/k)} \frac{q_n}{Q_n Q_{n-1}} \right) \frac{p_n P_{n-1}}{p_n} t_n \alpha_{n}^{(1/k)-1} \frac{Q_{n-1}}{P_{n-1}} \epsilon_n
\]

\[
= \beta_n^{1-(1/k)} \left( \frac{q_n}{Q_n Q_{n-1}} \right) \sum_{v=1}^{n-1} \sum_{v=1}^{v} \left\{ \frac{-q_v}{P_{v-1}} \epsilon_v + \frac{P_v Q_v \epsilon_v}{P_{v-1} P_v} + \frac{Q_v \Delta \epsilon_v}{P_v} \right\}
\]

\[
+ \frac{P_n q_n}{p_n Q_n} \alpha_{n}^{(1/k)-1} t_n \epsilon_n.
\]

Let us denote the above form of \( T_n \) by \( T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \).
By Minkowski's inequality, in order to prove the sufficiency, it is sufficient to show that $\sum_{n=1}^{\infty} |T_{n,r}|^k < \infty, r = 1, 2, 3, 4$. Applying Hölder's inequality,
CHARACTERIZATION ON SOME ABSOLUTE SUMMABILITY FACTORS ... 821

\[ = O(1) \sum_{v=1}^{m} \left( \frac{P_{v-1}}{P_v} \right)^k \left( \frac{\beta_v}{\alpha_v} \right)^{k-1} |t_v|^k |\Delta \epsilon_v|^k, \]

\[ \sum_{n=2}^{m+1} |T_{n,n}|^k = O(1) \sum_{n=1}^{m} \left( \frac{q_n P_n}{p_n Q_n} \right)^k \left( \frac{\beta_n}{\alpha_n} \right)^{k-1} |t_n|^k |\epsilon_n|^k. \]

(3.10)

Sufficiency of (3.6) and (3.7) follows.

**NECESSITY OF (3.6).** Using the result of Bor in [2], the transformation from \((t_n)\) into \((T_n)\) maps \(\ell^k\) into \(\ell^k\) and, hence by Lemma 2.1 the diagonal elements of this transformation are bounded and so (3.6) is necessary.

**NECESSITY OF (3.7).** This follows from Lemma 2.2 and the necessity of (3.6) by taking

\[ f_n = \left( \frac{p_n}{P_n} \right) \left( \frac{\alpha_n}{\beta_n} \right)^{1-(1/k)}, \quad g_n = \frac{Q_n}{q_n}, \]  

(3.11)

\[ \square \]

4. Applications

**COROLLARY 4.1.** Suppose that the conditions (3.1) and (3.2) are satisfied. Then the necessary and sufficient condition that \(\sum a_n\) be summable \(\text{\textsc{N}}, q_n, \beta_n|k, k \geq 1,\) is

\[ \frac{P_n q_n}{P_n Q_n} = O \left\{ \left( \frac{\alpha_n}{\beta_n} \right)^{1-(1/k)} \right\}. \]  

(4.1)

**PROOF.** The proof follows from Theorem 3.1 by putting \(\epsilon_n = 1\) and noticing that we do not need the conditions (3.3), (3.4), and (3.5) as \(\Delta \epsilon_n = 0\) for \(\epsilon_n = 1. \) \(\square \)

**COROLLARY 4.2.** Suppose that (3.2) and (3.4) are satisfied, \(\{(P_n q_n / p_n Q_n)^{(1/k)} \epsilon_n\}\) is monotonic, and

\[ \Delta \left\{ \frac{P_n \left( \frac{p_n q_n}{p_n Q_n} \right)^{1-(1/k)}}{P_n Q_n} \right\} = O \left\{ \frac{P_{n+1} q_{n+1}}{P_{n+1} Q_{n+1}} \left( \frac{P_n q_n}{P_n Q_n} \right)^{1-(1/k)} \right\}. \]  

(4.2)

Then the necessary and sufficient conditions that \(\sum a_n \epsilon_n\) be summable \(\text{\textsc{N}}, q_n|k, k \geq 1,\) are

\[ \epsilon_n = O \left\{ \frac{P_n Q_n}{p_n q_n} \right\}^{1/k}, \quad \Delta \epsilon_n = \left\{ \frac{p_n}{P_n-1} \left( \frac{P_n q_n}{p_n Q_n} \right)^{1-(1/k)} \right\}. \]  

(4.3)

**PROOF.** The proof follows from Theorem 3.1 by putting \(\alpha_n = P_n / p_n, \beta_n = Q_n / q_n. \) \(\square \)

**COROLLARY 4.3** (Bor and Thorpe [3]). Suppose that \(p_n Q_n = O(P_n q_n)\) and \(P_n q_n = O(p_n Q_n).\) Then, the series \(\sum a_n\) is summable \(\text{\textsc{N}}, q_n|k, k \geq 1,\) if and only if it is summable \(\text{\textsc{N}}, p_n|k, k \geq 1.\)

**PROOF.** The proof follows from the sufficient part of Corollary 4.1. \(\square \)
Remark. It may be noticed that (3.4) can be replaced by

\[ Q_n \Delta q_n = O(q_n q_{n+1}), \quad (4.4) \]

as

\[
\left| \Delta \left( \frac{Q_n}{q_n} \right) \right| = \left| \frac{Q_n}{q_n} - \frac{Q_{n+1}}{q_{n+1}} \right| = \left| \frac{q_{n+1} Q_n - q_n (Q_n + q_{n+1})}{q_n q_{n+1}} \right| \\
= \frac{Q_n \Delta q_n + 1}{q_n q_{n+1}} \quad (4.5) \\
\leq 1 + \frac{Q_n |\Delta q_n|}{q_n q_{n+1}}.
\]

References


Sulaiman: College of Education, Ajman University, P. O. Box 346, Ajman, United Arab Emirates
Mathematical Problems in Engineering

Special Issue on
Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>February 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>May 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>August 1, 2009</td>
</tr>
</tbody>
</table>

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; elbert@lac.inpe.br

Celso Grebogi, Department of Physics, King’s College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk

Hindawi Publishing Corporation
http://www.hindawi.com