ABEL-TYPE WEIGHTED MEANS TRANSFORMATIONS INTO $\ell$

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(Received 15 July 1999)

Abstract. Let $q_k = \binom{k+\alpha}{k}$ for $\alpha > -1$ and $Q_n = \sum_{k=0}^{n} q_k$. Suppose $A_q = \{a_{nk}\}$, where $a_{nk} = q_k/Q_n$ for $0 \leq k \leq n$ and 0 otherwise. $A_q$ is called the Abel-type weighted mean matrix. The purpose of this paper is to study these transformations as mappings into $\ell$. A necessary and sufficient condition for $A_q$ to be $\ell$-$\ell$ is proved. Also some other properties of the $A_q$ matrix are investigated.

Keywords and phrases. $\ell$-$\ell$ methods, G-G methods, $G_w$-$G_w$ methods.

2000 Mathematics Subject Classification. Primary 40A05, 40D25; Secondary 40C05.

1. Introduction. Throughout this paper, we assume that $\alpha > -1$ and $Q_n$ is the partial sums of the sequence $\{q_k\}$, where $q_k$ is as above. Let $A_q = \{a_{nk}\}$. Then the Abel-type weighted mean matrix, denoted by $A_q$, is defined by

$$a_{nk} = \begin{cases} 
q_k/Q_n & \text{for } 0 \leq k \leq n, \\
0 & \text{for } k > n.
\end{cases} \quad (1.1)$$

The $A_q$ matrix is the weighted mean matrix that is associated with the Abel-type matrix introduced by M. Lemma in [5]. It is regular, indeed, totally regular.

2. Basic notation and definitions. Let $A = (a_{nk})$ be an infinite matrix defining a sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, \quad (2.1)$$

where $(Ax)_n$ denotes the $n$th term of the image sequence $Ax$. Let $\gamma$ be a complex number sequence. Throughout this paper, we use the following basic notation and definitions:

(i) $c = \{\text{The set of all convergent complex sequences}\}$,
(ii) $\ell = \{\gamma : \sum_{k=0}^{\infty} |\gamma_k| < \infty\}$,
(iii) $\ell^p = \{\gamma : \sum_{k=0}^{\infty} |\gamma_k|^p < \infty\}$,
(iv) $\ell(A) = \{\gamma : Ay \in \ell\}$,
(v) $G = \{\gamma : \gamma_k = O(r^k) \text{ for some } r \in (0,1)\}$,
(vi) $G_w = \{\gamma : \gamma_k = O(r^k) \text{ for some } r \in (0,w), \ 0 < w < 1\}$. 

DEFINITION 1. If $X$ and $Y$ are sets of complex number sequences, then the matrix $A$ is called an $X$-$Y$ matrix if the image $Au$ of $u$ under the transformation $A$ is in $Y$, whenever $u$ is in $X$.

3. Some basic facts. The following facts are used repeatedly.

(1) For any real number $\alpha > -1$ and any nonnegative integer $k$, we have
\[
\binom{k+\alpha}{k} \sim \frac{k^\alpha}{\Gamma(\alpha+1)} \quad \text{as } k \to \infty.
\] (3.1)

(2) For any real number $\alpha > -1$, we have
\[
\sum_{k=0}^{n} \binom{k+\alpha}{k} = \binom{n+\alpha+1}{n}.
\] (3.2)

(3) Suppose $\{a_n\}$ is sequence of nonnegative numbers with $a_0 > 0$, that
\[
A_n = \sum_{k=0}^{n} a_k \to \infty.
\] (3.3)

Let
\[
a(x) = \sum_{k=0}^{\infty} a_k x^k, \quad A(x) = \sum_{k=0}^{\infty} A_k x^k,
\] (3.4)

and suppose that
\[
a(x) < \infty \quad \text{for } 0 < x < 1.
\] (3.5)

Then it follows that
\[
(1-x)A(x) = a(x) \quad \text{for } 0 < x < 1.
\] (3.6)

4. The main results

Lemma 1. If $A_q$ is an $\ell$-$\ell$ matrix, then $1/Q \in \ell$.

Proof. By the Knopp-Lorentz theorem [4], $A_q$ is an $\ell$-$\ell$ matrix implies that
\[
\sum_{k=0}^{\infty} |a_{n,0}| < \infty,
\] (4.1)

and consequently we have $1/Q \in \ell$. \qed

Lemma 2. We have that $1/Q \in \ell$ if and only if $\alpha > 0$.

Proof. By using (3.1), we have
\[
\frac{1}{Q_n} \sim \frac{\Gamma(\alpha+2)}{n^{\alpha+1}}
\] (4.2)

and hence the assertion easily follows. \qed
**Lemma 3.** If $1/Q \in \ell$, then $A_q$ is an $\ell$-$\ell$ matrix.

**Proof.** By Lemma 2, we have $\alpha > 0$. To show that $A_q$ is an $\ell$-$\ell$ matrix, we must show that the condition of the Knopp-Lorentz theorem [4] holds. Using (3.1), we have

$$
\sum_{n=0}^{\infty} |a_{nk}| = \left(\frac{k+\alpha}{k}\right) \sum_{n=k}^{\infty} \frac{1}{Q_n} = \left(\frac{k+\alpha}{k}\right) \sum_{n=k}^{\infty} \frac{1}{n^{n+\alpha+1}} \\
\leq M_1 K^\alpha \sum_{n=k}^{\infty} \frac{1}{n^{\alpha+1}} \quad \text{for some } M_1 > 0,
$$

(4.3)

Hence, by the Knopp-Lorentz theorem [4], $A_q$ is an $\ell$-$\ell$ matrix. $\square$

**Theorem 1.** The following statements are equivalent:

1. $A_q$ is an $\ell$-$\ell$ matrix;
2. $1/Q \in \ell$;
3. $\alpha > 0$.

**Proof.** The theorem easily follows by Lemmas 1, 2, and 3. $\square$

**Remark 1.** In Theorem 1, we showed that $A_q$ is an $\ell$-$\ell$ matrix if and only if $1/Q \in \ell$. But the converse is not true in general for any weighted mean matrix $W_p$ that corresponds to a sequence-to-sequence variant of the general $J_p$ power series method of summability [1]. To see this, let

$$
p_k = (\ln(k+2))^{\alpha}, \quad \alpha > 1.
$$

(4.4)

We show that $1/P \in \ell$ but $W_p$ is not an $\ell$-$\ell$ matrix. We have

$$
P_n = \sum_{k=0}^{n} (\ln(k+2))^{\alpha} \\
\sim \int_{0}^{n} (\ln(x+2))^{\alpha} dx \quad \text{(by [6, Thm. 1.20])}
$$

(4.5)

Using integration by parts repeatedly. This yields

$$
\frac{1}{P_n} \sim \frac{1}{(n+2)(\ln(n+2))^{\alpha}}
$$

(4.6)

and by the condensation test, it follows that $1/P \in \ell$. 
Next, we show that $W_p$ is not an $ℓ$-$ℓ$ matrix by showing that the condition of the Knopp-Lorentz theorem [4] fails to hold. Using (4.6), it follows that
\[
\sum_{n=0}^{∞} |a_{nk}| = (\ln(k + 2))^α \sum_{n=k}^{∞} \frac{1}{p_n}
\geq M_1 (\ln(k + 2))^α \sum_{n=k}^{∞} \frac{1}{(n+2)(\ln(n+2))^α} \quad \text{for some } M_1 > 0
\]
\[
\geq M_1 M_2 (\ln(k + 2))^α \int_{k}^{∞} \frac{dx}{(x+2)(\ln(x+2))^α} \quad \text{for some } M_2 > 0
\]
\[
= \frac{M_1 M_2}{α-1} (\ln(k + 2)).
\]

Thus, we have
\[
\sup_k \left\{ \sum_{n=0}^{∞} |a_{nk}| \right\} = ∞,
\]
and hence $W_p$ is not an $ℓ$-$ℓ$ matrix. □

**Corollary 1.** $A_q$ is an $ℓ$-$ℓ$ matrix.

**Proof.** Since $Q_n = \left(\frac{n+α+1}{n}\right)$ and $α > -1$ implies that $α + 1 > 0$, the assertion easily follows by Theorem 1. □

**Corollary 2.** $A_q$ is an $ℓ$-$ℓ$ matrix if and only if $\lim_n (Q_n/nq_n) < 1$.

**Proof.** By Theorem 1, $A_q$ is an $ℓ$-$ℓ$ matrix implies that $α > 0$, and as a consequence we have $1/(α + 1) < 1$. Now using (3.1), we have
\[
\lim_n \left( \frac{Q_n}{nq_n} \right) = \lim_n \frac{n^{α+1}\Gamma(α+1)}{Γ(α+1)n^{α+1}} = \frac{1}{α+1} < 1.
\]
Conversely, if $\lim_n (Q_n/nq_n) < 1$, then it follows from (4.9) that $1/(α + 1) < 1$ and consequently we have $α > 0$, and hence, by Theorem 1, $A_q$ is an $ℓ$-$ℓ$ matrix. □

**Corollary 3.** Suppose that $z_k = \left(\frac{k+β}{k}\right)$ and $α < β$; then $A_z$ is an $ℓ$-$ℓ$ matrix whenever $A_q$ is an $ℓ$-$ℓ$ matrix.

**Proof.** The corollary follows easily by Theorem 1. □

**Lemma 4.** If the Abel-type matrix $A_{α,t}$ [5] is an $ℓ$-$ℓ$ matrix, then $A_{α+1,t}$ is also an $ℓ$-$ℓ$ matrix.

**Proof.** By the Knopp-Lorentz theorem [4], $A_{α,t}$ is an $ℓ$-$ℓ$ matrix implies that
\[
\sup_k \left\{ \sum_{n=0}^{∞} |a_{nk}| \right\} < ∞.
\]
This is equivalent to
\[
\sup_k \left\{ \left(\frac{k+α}{k}\right) \sum_{n=0}^{∞} t_n^k (1-t_n)^{α+1} \right\} < ∞.
\]
Now from (4.11), we can easily conclude that

$$\sup_k \left\{ \left( \frac{k+\alpha+1}{k} \right) \sum_{n=0}^{\infty} t_n^k (1-t_n)^{\alpha+2} \right\} < \infty. \quad (4.12)$$

Hence, $A_{\alpha+1,t}$ is an $\ell$-$\ell$ matrix.

The next theorem compares the summability fields of the matrices $A_q$ and $A_{\alpha,t}$ [5].

**Theorem 2.** If $A_{\alpha,t}$ and $A_q$ are $\ell$-$\ell$ matrices, then $\ell(A_q) \subseteq \ell(A_{\alpha,t})$.

**Proof.** Let $x \in \ell(A_q)$. Then we show that $x \in \ell(A_{\alpha,t})$. Let $y$ be the $A_q$-transform of the sequence $x$. Then we have

$$y_n Q_n = \sum_{k=0}^{n} q_k x_k. \quad (4.13)$$

Now since $y_n Q_n$ is the partial sums of the sequence $q_x$, using (3.6) it follows that

$$(1-t_n) \sum_{k=0}^{\infty} Q_k y_k t_n^k = \sum_{k=0}^{\infty} q_k x_k t_n^k. \quad (4.14)$$

This yields

$$(1-t_n)^{\alpha+2} \sum_{k=0}^{\infty} Q_k y_k t_n^k = (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} q_k x_k t_n^k, \quad (4.15)$$

and as a consequence we have $(A_{\alpha+1,t} y)_n = (A_{\alpha,t} x)_n$. By Lemma 4, $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix implies that $A_{\alpha+1,t}$ is also an $\ell$-$\ell$ matrix, and from the assumption that $x \in \ell(A_q)$, it follows that $y \in \ell$. Consequently, we have $A_{\alpha+1,t} y \in \ell$ and this is equivalent to $A_{\alpha,t} x \in \ell$. Thus, $x \in \ell(A_{\alpha,t})$ and hence our assertion follows.

**Remark 2.** Theorem 2 gives an important inclusion result in the $\ell$-$\ell$ setting that parallels the famous inclusion result that exists between the power series method of summability and its corresponding weighted mean in the $c$-$c$ setting [1].

**Lemma 5.** Suppose $A = \{a_{nk}\}$ is an $\ell$-$\ell$ matrix such that $a_{nk} = 0$ for $k > n, m > s$ (both positive integers); then $\ell(A^s) \subseteq \ell(A^m)$, where the interpretation for $A^s$ and $A^m$ is as given in [6, p. 28].

**Theorem 3.** If $B = A_q$ is an $\ell$-$\ell$ matrix, then $B^m$ is also an $\ell$-$\ell$ matrix (for $m$ a positive integer greater than 1.)

**Proof.** Let $x \in \ell, B$ is an $\ell$-$\ell$ matrix implies that $x \in \ell(B)$. By Lemma 5, we have $\ell(B) \subseteq \ell(B^m)$ and hence it follows that $x \in \ell(B^m)$. Hence, $B^m$ is an $\ell$-$\ell$ matrix.

**Remark 3.** Theorem 3 gives a result that goes parallel to a c-$c$ result given on [6, Thm. 2.4, p. 28].

In Corollary 1, we showed that $A_Q$ is an $\ell$-$\ell$ matrix. Here, a question may be raised as to whether $A_Q$ maps $\ell^p$ into $\ell$ for $p > 1$. But this is answered negatively by the following theorem.
**Theorem 4.** $A_Q$ does not map $\ell^p$ into $\ell$ for $p > 1$.

**Proof.** Let $A_Q = \{b_{nk}\}$. Note that if $A_{Q,\alpha}$ maps $\ell^p$ into $\ell$, then by [3, Thm. 2], we must have
\[
\lim_{k} \sum_{n=1}^{\infty} |b_{nk}| = 0. \tag{4.16}
\]
Let
\[
R_n = \sum_{k=1}^{n} Q_k, \tag{4.17}
\]
then it follows that
\[
\sum_{n=1}^{\infty} b_{nk} = \binom{k + \alpha + 1}{k} \sum_{n=1}^{\infty} \frac{1}{R_n} = \binom{k + \alpha + 1}{k} \sum_{n=k}^{\infty} \frac{1}{n^{\alpha+2}} \geq M_1 k^{\alpha+1} \sum_{n=k}^{\infty} \frac{1}{n^{\alpha+2}} \text{ for some } M_1 > 0 \tag{4.18}
\]
\[
\geq M_1 M_2 k^{\alpha+1} \int_{k}^{\infty} \frac{dx}{x^{\alpha+2}} \text{ for some } M_2 > 0
\]
\[
= \frac{M_1 M_2}{\alpha+1} > 0.
\]
Thus, it follows that
\[
\lim_{k} \sum_{n=1}^{\infty} |b_{nk}| > 0, \tag{4.19}
\]
and hence $A_Q$ does not map $\ell^p$ into $\ell$ for $p > 1$ by [3, Thm. 2].

Our next theorem has the form of an extension mapping theorem. It indicates that a mapping of $A_q$ from $G$ or $G_w$ into $\ell$ can be extended to a mapping of $\ell$ into $\ell$.

**Theorem 5.** The following statements are equivalent:
\begin{enumerate}
  \item $A_q$ is an $\ell$-$\ell$ matrix;
  \item $A_q$ is a $G$-$\ell$ matrix;
  \item $A_q$ is a $G_w$-$\ell$ matrix.
\end{enumerate}

**Proof.** Since $G$ is a subset of $\ell$ and $G_w$ a subset of $G$, (1)$\Rightarrow$(2)$\Rightarrow$(3) follow easily. The assertion that (3)$\Rightarrow$(1) follows by [7, Thm. 1.1] and Theorem 1.

**Corollary 4.** (1) $A_Q$ is a $G$-$\ell$ matrix.
(2) $A_Q$ a $G_w$-$\ell$ matrix.

**Proof.** Since $A_Q$ is an $\ell$-$\ell$ matrix by Corollary 1, the assertion follows by Theorem 5.

**Corollary 5.** (1) If $A_q$ is a $G$-$G$ matrix, then $A_q$ is an $\ell$-$\ell$ matrix.
(2) If $A_q$ is a $G_w$-$G_w$ matrix, then $A_q$ is an $\ell$-$\ell$ matrix.

**Proof.** The assertion follows easily by Theorem 5.
Theorem 6. \( A_q \) is a \( G-G \) matrix if and only if \( 1/Q \in G \).

Proof. If \( A_q \) is a \( G-G \) matrix, then the first column of \( A_q \) is must in \( G \). This gives \( 1/Q \in G \) since \( a_{n,0} = q_0 / Q_n \). Conversely, suppose \( 1/Q \in G \). Then \( 1/Q_n \leq M_1 r^n \) for \( M_1 > 0 \) and \( r \in (0,1) \). Now let \( u \in G \), say \( |u_k| \leq M_2 t^k \) for some \( M_2 > 0 \) and \( t \in (0,1) \). Let \( Y \) be the \( A_q \)-transform of the sequence \( u \). Then we have

\[
|Y_n| \leq M_1 M_2 r^n \sum_{k=0}^{n} \binom{k+\alpha}{k} t^k < M_1 M_2 r^n (1-t)^{-\alpha-1} < M_3 r^n \quad \text{for some } M_3 > 0.
\]

(4.20)

Therefore, \( Y \in G \) and hence it follows that \( A_q \) is a \( G-G \) matrix.

Theorem 7. \( A_q \) is a \( G_w-G_w \) matrix if and only if \( 1/Q \in G_w \).

Proof. The proof follows easily using the same steps as in the proof of Theorem 6 by replacing \( G \) with \( G_w \).

Lemma 6. If the Abel-type matrix \( A_{\alpha,t} \) [5] is a \( G-G \) matrix, then \( A_{\alpha+1,t} \) is also a \( G-G \) matrix.

Proof. By [5, Thm. 7], \( A_{\alpha,t} \) is \( G-G \) implies that \( (1-t)^{\alpha+1} \in G \). But \( (1-t)^{\alpha+1} \in G \) yields \( (1-t)^{\alpha+2} \in G \), and hence by [5, Thm. 7], it follows that \( A_{\alpha+1,t} \) is a \( G-G \) matrix.

Lemma 7. If the Abel-type matrix \( A_{\alpha,t} \) [5] is a \( G_w-G_w \) matrix, then \( A_{\alpha+1,t} \) is also a \( G_w-G_w \) matrix.

Proof. The assertion easily follows by replacing \( G \) with \( G_w \) in the proof of Lemma 6.

Theorem 8. If \( A_{\alpha,t} \) [5] and \( A_q \) are \( G-G \) matrices, then the \( G(A_{\alpha,t}) \) contains \( G(A_q) \).

Proof. The proof easily follows using the same techniques as in the proof of Theorem 3 by replacing \( \ell \) with \( G \) and applying Lemma 6.

Theorem 9. If \( A_{\alpha,t} \) [3] and \( A_q \) are \( G_w-G_w \) matrices, then \( G_w(A_{\alpha,t}) \) contains \( G_w(A_q) \).

Proof. The proof easily follows using the same techniques as in the proof of Theorem 3 by replacing \( \ell \) with \( G_w \) and applying Lemma 7.

References


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