INTUITIONISTIC FUZZY IDEALS OF BCK-ALGEBRAS

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Abstract. We consider the intuitionistic fuzzification of the concept of subalgebras and ideals in BCK-algebras, and investigate some of their properties. We introduce the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a BCK-algebra and investigate some related properties.

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1. Introduction. After the introduction of the concept of fuzzy sets by Zadeh [9] several researches were conducted on the generalizations of the notion of fuzzy sets. The idea of “intuitionistic fuzzy set” was first published by Atanassov [1, 2], as a generalization of the notion of fuzzy set. The first author (together with Hong, Kim, Kim, Meng, Roh, and Song) considered the fuzzification of ideals and subalgebras in BCK-algebras (cf. [3, 4, 5, 6, 7, 8]). In this paper, using the Atanassov’s idea, we establish the intuitionistic fuzzification of the concept of subalgebras and ideals in BCK-algebras, and investigate some of their properties. We introduce the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a BCK-algebra and investigate some related properties.

2. Preliminaries. First we present the fundamental definitions. By a $BCK$-algebra we mean a nonempty set $X$ with a binary operation $\ast$ and a constant $0$ satisfying the following conditions:

(I) \((x \ast y) \ast (x \ast z) \ast (z \ast y) = 0,\)

(II) \((x \ast (x \ast y)) \ast y = 0,\)

(III) \(x \ast x = 0,\)

(IV) \(0 \ast x = 0,\)

(V) \(x \ast y = 0\) and \(y \ast x = 0\) imply that \(x = y\)

for all \(x, y, z \in X.\)

A partial ordering “$\leq$” on $X$ can be defined by \(x \leq y\) if and only if \(x \ast y = 0.\)

A nonempty subset $S$ of a $BCK$-algebra $X$ is called a subalgebra of $X$ if \(x \ast y \in S\) whenever \(x, y \in S.\) A nonempty subset $I$ of a $BCK$-algebra $X$ is called an ideal of $X$ if

(i) \(0 \in I,\)

(ii) \(x \ast y \in I\) and \(y \in I\) imply that \(x \in I\) for all \(x, y \in X.\)

By a fuzzy set \(\mu\) in a nonempty set $X$ we mean a function \(\mu : X \rightarrow [0, 1],\) and the complement of $\mu$, denoted by $\bar{\mu}$, is the fuzzy set in $X$ given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$. A fuzzy set $\mu$ in a $BCK$-algebra $X$ is called a fuzzy subalgebra of $X$ if $\mu(x \ast y) \geq
min\{\mu(x),\mu(y)\} for all \(x, y \in X\). A fuzzy set \(\mu\) in a BCK-algebra \(X\) is called a fuzzy ideal of \(X\) if

(i) \(\mu(0) \geq \mu(x)\) for all \(x \in X\),

(ii) \(\mu(x) \geq \min\{\mu(x*y),\mu(y)\}\) for all \(x, y \in X\).

An intuitionistic fuzzy set (briefly, IFS) \(A\) in a nonempty set \(X\) is an object having the form

\[A = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\},\] (2.1)

where the functions \(\alpha_A : X \to [0, 1]\) and \(\beta_A : X \to [0, 1]\) denote the degree of membership and the degree of nonmembership, respectively, and

\[0 \leq \alpha_A(x) + \beta_A(x) \leq 1 \quad \forall x \in X.\] (2.2)

An intuitionistic fuzzy set \(A = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\}\) in \(X\) can be identified to an ordered pair \((\alpha_A, \beta_A)\) in \(I^X \times I^X\). For the sake of simplicity, we shall use the symbol \(A = (\alpha_A, \beta_A)\) for the IFS \(A = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\}\).

### 3. Intuitionistic fuzzy ideals.

In what follows, let \(X\) denote a BCK-algebra unless otherwise specified.

**Definition 3.1.** An IFS \((\alpha_A, \beta_A)\) in \(X\) is called an intuitionistic fuzzy subalgebra of \(X\) if it satisfies:

(IS1) \(\alpha_A(x*y) \geq \min\{\alpha_A(x), \alpha_A(y)\}\),

(IS2) \(\beta_A(x*y) \leq \max\{\beta_A(x), \beta_A(y)\}\),

for all \(x, y \in X\).

**Example 3.2.** Consider a BCK-algebra \(X = \{0, a, b, c\}\) with the following Cayley table:

<table>
<thead>
<tr>
<th>(*)</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Let \(A = (\alpha_A, \beta_A)\) be an IFS in \(X\) defined by

\[
\begin{align*}
\alpha_A(0) &= \alpha_A(a) = \alpha_A(c) = 0.7 > 0.3 = \alpha_A(b), \\
\beta_A(0) &= \beta_A(a) = \beta_A(c) = 0.2 < 0.5 = \beta_A(b).
\end{align*}
\] (3.1)

Then \(A = (\alpha_A, \beta_A)\) is an intuitionistic fuzzy subalgebra of \(X\).

**Proposition 3.3.** Every intuitionistic fuzzy subalgebra \(A = (\alpha_A, \beta_A)\) of \(X\) satisfies the inequalities \(\alpha_A(0) \geq \alpha_A(x)\) and \(\beta_A(0) \leq \beta_A(x)\) for all \(x \in X\).

**Proof.** For any \(x \in X\), we have

\[
\begin{align*}
\alpha_A(0) &= \alpha_A(x*y) \geq \min\{\alpha_A(x), \alpha_A(x)\} = \alpha_A(x), \\
\beta_A(0) &= \beta_A(x*y) \leq \max\{\beta_A(x), \beta_A(x)\} = \beta_A(x).
\end{align*}
\] (3.2)

This completes the proof. \qed
**Definition 3.4.** An IFS \( A = (\alpha_A, \beta_A) \) in \( X \) is called an *intuitionistic fuzzy ideal* of \( X \) if it satisfies the following inequalities:

1. \( \alpha_A(0) \geq \alpha_A(x) \) and \( \beta_A(0) \leq \beta_A(x) \),
2. \( \alpha_A(x) \geq \min\{\alpha_A(x \ast y), \alpha_A(y)\} \),
3. \( \beta_A(x) \leq \max\{\beta_A(x \ast y), \beta_A(y)\} \),

for all \( x, y \in X \).

**Example 3.5.** Let \( X = \{0, 1, 2, 3, 4\} \) be a BCK-algebra with the following Cayley table:

\[
\begin{array}{c|cccc}
* & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
2 & 2 & 2 & 0 & 0 \\
3 & 3 & 3 & 3 & 0 \\
4 & 4 & 3 & 4 & 1 \\
\end{array}
\]

Define an IFS \( A = (\alpha_A, \beta_A) \) in \( X \) as follows:

\[
\begin{align*}
\alpha_A(0) &= 1, & \alpha_A(1) &= \alpha_A(3) = \alpha_A(4) = t, \\
\beta_A(0) &= 0, & \beta_A(1) &= \beta_A(3) = \beta_A(4) = s,
\end{align*}
\]

where \( t \in [0, 1], s \in [0, 1], \) and \( t + s \leq 1 \). By routine calculation we know that \( A = (\alpha_A, \beta_A) \) is an intuitionistic fuzzy ideal of \( X \).

**Lemma 3.6.** Let an IFS \( A = (\alpha_A, \beta_A) \) in \( X \) be an intuitionistic fuzzy ideal of \( X \). If the inequality \( x \ast y \leq z \) holds in \( X \), then

\[
\begin{align*}
\alpha_A(x) &\geq \min \{\alpha_A(y), \alpha_A(z)\}, & \beta_A(x) &\leq \max \{\beta_A(y), \beta_A(z)\}. 
\end{align*}
\]

**Proof.** Let \( x, y, z \in X \) be such that \( x \ast y \leq z \). Then \( (x \ast y) \ast z = 0 \), and thus

\[
\begin{align*}
\alpha_A(x) &\geq \min \{\alpha_A(x \ast y), \alpha_A(y)\} \\
&\geq \min \{\min \{\alpha_A((x \ast y) \ast z), \alpha_A(z)\}, \alpha_A(y)\} \\
&= \min \{\min \{\alpha_A(0), \alpha_A(z)\}, \alpha_A(y)\} \\
&= \min \{\alpha_A(y), \alpha_A(z)\},
\end{align*}
\]

\[
\begin{align*}
\beta_A(x) &\leq \max \{\beta_A(x \ast y), \beta_A(y)\} \\
&\leq \max \{\max \{\beta_A((x \ast y) \ast z), \beta_A(z)\}, \beta_A(y)\} \\
&= \max \{\max \{\beta_A(0), \beta_A(z)\}, \beta_A(y)\} \\
&= \max \{\beta_A(y), \beta_A(z)\},
\end{align*}
\]

this completes the proof.

**Lemma 3.7.** Let \( A = (\alpha_A, \beta_A) \) be an intuitionistic fuzzy ideal of \( X \). If \( x \leq y \) in \( X \), then

\[
\begin{align*}
\alpha_A(x) &\geq \alpha_A(y), & \beta_A(x) &\leq \beta_A(y), 
\end{align*}
\]

that is, \( \alpha_A \) is order-preserving and \( \beta_A \) is order-preserving.
Proof. Let \( x, y \in X \) be such that \( x \leq y \). Then \( x \ast y = 0 \) and so
\[
\alpha_A(x) \geq \min \{ \alpha_A(x \ast y), \alpha_A(y) \} = \min \{ \alpha_A(0), \alpha_A(y) \} = \alpha_A(y),
\]
\[
\beta_A(x) \leq \max \{ \beta_A(x \ast y), \beta_A(y) \} = \max \{ \beta_A(0), \beta_A(y) \} = \beta_A(y).
\] (3.7)
This completes the proof. □

Theorem 3.8. If \( A = (\alpha_A, \beta_A) \) is an intuitionistic fuzzy ideal of \( X \), then for any \( x, a_1, a_2, \ldots, a_n \in X \), \((\cdots ((x \ast a_1) \ast a_2) \ast \cdots) \ast a_n = 0\) implies
\[
\alpha_A(x \ast y) \geq \min \{ \alpha_A(a_1), \alpha_A(a_2), \ldots, \alpha_A(a_n) \},
\]
\[
\beta_A(x \ast y) \leq \max \{ \beta_A(a_1), \beta_A(a_2), \ldots, \beta_A(a_n) \}. \] (3.8)

Proof. Using induction on \( n \) and Lemmas 3.6 and 3.7, the proof is straightforward. □

Theorem 3.9. Every intuitionistic fuzzy ideal of \( X \) is an intuitionistic fuzzy subalgebra of \( X \).

Proof. Let \( A = (\alpha_A, \beta_A) \) be an intuitionistic fuzzy ideal of \( X \). Since \( x \ast y \leq x \) for all \( x, y \in X \), it follows from Lemma 3.7 that
\[
\alpha_A(x \ast y) \geq \alpha_A(x), \quad \beta_A(x \ast y) \leq \beta_A(x), \] (3.9)
so by (IF2) and (IF3),
\[
\alpha_A(x \ast y) \geq \alpha_A(x) \geq \min \{ \alpha_A(x \ast y), \alpha_A(y) \} \geq \min \{ \alpha_A(x), \alpha_A(y) \},
\]
\[
\beta_A(x \ast y) \leq \beta_A(x) \leq \max \{ \beta_A(x \ast y), \beta_A(y) \} \leq \max \{ \beta_A(x), \beta_A(y) \}. \] (3.10)
This shows that \( A = (\alpha_A, \beta_A) \) is an intuitionistic fuzzy subalgebra of \( X \). □

The converse of Theorem 3.9 may not be true. For example, the intuitionistic fuzzy subalgebra \( A = (\alpha_A, \beta_A) \) in Example 3.2 is not an intuitionistic fuzzy ideal of \( X \) since
\[
\beta_A(b) = 0.5 > 0.2 = \min \{ \beta_A(b \ast a), \beta_A(a) \}. \] (3.11)

We now give a condition for an intuitionistic fuzzy subalgebra to be an intuitionistic fuzzy ideal.

Theorem 3.10. Let \( A = (\alpha_A, \beta_A) \) be an intuitionistic fuzzy subalgebra of \( X \) such that
\[
\alpha_A(x) \geq \min \{ \alpha_A(y), \alpha_A(z) \}, \quad \beta_A(x) \leq \max \{ \beta_A(y), \beta_A(z) \} \] (3.12)
for all \( x, y, z \in X \) satisfying the inequality \( x \ast y \leq z \). Then \( A = (\alpha_A, \beta_A) \) is an intuitionistic fuzzy ideal of \( X \).

Proof. Let \( A = (\alpha_A, \beta_A) \) be an intuitionistic fuzzy subalgebra of \( X \). Recall that \( \alpha_A(0) \geq \alpha_A(x) \) and \( \beta_A(0) \leq \beta_A(x) \) for all \( X \). Since \( x \ast (x \ast y) \leq y \), it follows from the hypothesis that
\[
\alpha_A(x) \geq \min \{ \alpha_A(x \ast y), \alpha_A(y) \}, \quad \beta_A(x) \leq \max \{ \beta_A(x \ast y), \beta_A(y) \}. \] (3.13)
Hence \( A = (\alpha_A, \beta_A) \) is an intuitionistic fuzzy ideal of \( X \). □
**Lemma 3.11.** An IFS $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $X$ if and only if the fuzzy sets $\alpha_A$ and $\beta_A$ are fuzzy ideals of $X$.

**Proof.** Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy ideal of $X$. Clearly, $\alpha_A$ is a fuzzy ideal of $X$. For every $x, y \in X$, we have

\[
\hat{\beta}_A(0) = 1 - \beta_A(0) \geq 1 - \beta_A(x) = \hat{\beta}_A(x), \\
\hat{\beta}_A(x) = 1 - \beta_A(x) \geq 1 - \max \{\beta_A(x \ast y), \beta_A(y)\} \\
= \min \{1 - \beta_A(x \ast y), 1 - \beta_A(y)\} \\
= \min \{\hat{\beta}_A(x \ast y), \hat{\beta}_A(y)\}. \tag{3.14}
\]

Hence $\beta_A$ is a fuzzy ideal of $X$.

Conversely, assume that $\alpha_A$ and $\beta_A$ are fuzzy ideals of $X$. For every $x, y \in X$, we get

\[
\alpha_A(0) \geq \alpha_A(x), \\
1 - \beta_A(0) = \hat{\beta}_A(0) \geq \hat{\beta}_A(x) = 1 - \beta_A(x), \tag{3.15}
\]

that is, $\beta_A(0) \leq \beta_A(x)$; $\alpha_A(x) \leq \min \{\alpha_A(x \ast y), \alpha_A(y)\}$ and

\[
1 - \beta_A(x) = \hat{\beta}_A(x) \geq \min \{\hat{\beta}_A(x \ast y), \hat{\beta}_A(y)\} \\
= \min \{1 - \beta_A(x \ast y), 1 - \beta_A(y)\} \\
= 1 - \max \{\beta_A(x \ast y), \beta_A(y)\}, \tag{3.16}
\]

that is, $\beta_A(x) \leq \max \{\beta_A(x \ast y), \beta_A(y)\}$. Hence $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $X$. \qed

**Theorem 3.12.** Let $A = (\alpha_A, \beta_A)$ be an IFS in $X$. Then $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $X$ if and only if $\square A = (\alpha_A, \check{\alpha}_A)$ and $\Diamond A = (\hat{\beta}_A, \beta_A)$ are intuitionistic fuzzy ideals of $X$.

**Proof.** If $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $X$, then $\alpha_A = \check{\alpha}_A$ and $\beta_A$ are fuzzy ideals of $X$ from Lemma 3.11, hence $\square A = (\alpha_A, \check{\alpha}_A)$ and $\Diamond A = (\hat{\beta}_A, \beta_A)$ are intuitionistic fuzzy ideals of $X$. Conversely, if $\square A = (\alpha_A, \check{\alpha}_A)$ and $\Diamond A = (\hat{\beta}_A, \beta_A)$ are intuitionistic fuzzy ideals of $X$, then the fuzzy sets $\alpha_A$ and $\beta_A$ are fuzzy ideals of $X$, hence $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $X$. \qed

For any $t \in [0, 1]$ and a fuzzy set $\mu$ in a nonempty set $X$, the set

\[
U(\mu; t) = \{x \in X \mid \mu(x) \geq t\} \tag{3.17}
\]

is called an upper $t$-level cut of $\mu$ and the set

\[
L(\mu; t) = \{x \in X \mid \mu(x) \leq t\} \tag{3.18}
\]

is called a lower $t$-level cut of $\mu$.

**Theorem 3.13.** An IFS $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $X$ if and only if for all $s, t \in [0, 1]$, the sets $U(\alpha_A; t)$ and $L(\beta_A; s)$ are either empty or ideals of $X$. 

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PROOF. Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy ideal of $X$ and $U(\alpha_A; t) \neq \emptyset \neq L(\beta_A; s)$ for any $s,t \in [0,1]$. It is clear that $0 \in U(\alpha_A; t) \cap L(\beta_A; s)$ since $\alpha_A(0) \geq t$ and $\beta_A(0) \leq s$. Let $x, y \in X$ be such that $x * y \in U(\alpha_A; t)$ and $y \in U(\alpha_A; t)$. Then $\alpha_A(x * y) \geq t$ and $\alpha_A(y) \geq t$. It follows that

$$\alpha_A(x) \geq \min \{ \alpha_A(x * y), \alpha_A(y) \} \geq t \tag{3.19}$$

so that $x \in U(\alpha_A; t)$. Hence $U(\alpha_A; t)$ is an ideal of $X$. Now let $x, y \in X$ be such that $x * y \in L(\beta_A; s)$ and $y \in L(\beta_A; s)$. Then $\beta_A(x * y) \leq s$ and $\beta_A(y) \leq s$, which imply that

$$\beta_A(x) \leq \max \{ \beta_A(x * y), \beta_A(y) \} \leq s. \tag{3.20}$$

Thus $x \in L(\beta_A; s)$, and therefore $L(\beta_A; s)$ is an ideal of $X$. Conversely, assume that for each $t,s \in [0,1]$, the sets $U(\alpha_A; t)$ and $L(\beta_A; s)$ are either empty or ideals of $X$. For any $x \in X$, let $\alpha_A(x) = t$ and $\beta_A(x) = s$. Then $x \in U(\alpha_A; t) \cap L(\beta_A; s)$, and so $U(\alpha_A; t) \neq \emptyset \neq L(\beta_A; s)$. Since $U(\alpha_A; t)$ and $L(\beta_A; s)$ are ideals of $X$, therefore $0 \in U(\alpha_A; t) \cap L(\beta_A; s)$. Hence $\alpha_A(0) \geq t = \alpha_A(x)$ and $\beta_A(0) \leq s = \beta_A(x)$ for all $x \in X$. If there exist $x',y' \in X$ such that $\alpha_A(x') < \min \{ \alpha_A(x' * y'), \alpha_A(y') \}$, then by taking

$$t_0 = \frac{1}{2} (\alpha_A(x') + \min \{ \alpha_A(x' * y'), \alpha_A(y') \}), \tag{3.21}$$

we have

$$\alpha_A(x') < t_0 < \min \{ \alpha_A(x' * y'), \alpha_A(y') \}. \tag{3.22}$$

Hence $x' \notin U(\alpha_A; t_0)$, $x' * y' \in U(\alpha_A; t_0)$ and $y' \in (\alpha_A; t_0)$, that is, $U(\alpha_A; t_0)$ is not an ideal of $X$, which is a contradiction. Finally, assume that there exist $a, b \in X$ such that

$$\beta_A(a) \geq \max \{ \beta_A(a * b), \beta_A(b) \}. \tag{3.23}$$

Taking $s_0 := (1/2)(\beta_A(a) + \max \{ \beta_A(a * b), \beta_A(b) \})$, then

$$\max \{ \beta_A(a * b), \beta_A(b) \} < s_0 < \beta_A(a). \tag{3.24}$$

Therefore $a * b \notin L(\beta_A; s_0)$ and $b \in L(\beta_A; s_0)$, but $a \notin L(\beta_A; s_0)$, which is a contradiction, this completes the proof. \qed

Let $\Lambda$ be a nonempty subset of $[0,1]$.

**Theorem 3.14.** Let $\{I_t \mid t \in \Lambda\}$ be a collection of ideals of $X$ such that

(i) $X = \cup_{t \in \Lambda} I_t$,

(ii) $s > t$ if and only if $I_s \subset I_t$ for all $s,t \in \Lambda$.

Then an IFS $A = (\alpha_A, \beta_A)$ in $X$ defined by

$$\alpha_A(x) := \sup \{ t \in \Lambda \mid x \in I_t \}, \quad \beta_A(x) := \inf \{ t \in \Lambda \mid x \in I_t \} \tag{3.25}$$

for all $x \in X$ is an intuitionistic fuzzy ideal of $X$.

**Proof.** According to Theorem 3.13, it is sufficient to show that $U(\alpha_A; t)$ and $L(\beta_A; s)$ are ideals of $X$ for every $t \in [0, \alpha_A(0)]$ and $s \in [\beta_A(0), 1]$. In order to prove
that \( U(\alpha_A; t) \) is an ideal of \( X \), we divide the proof into the following two cases:

(i) \( t = \sup \{ q \in \Lambda \mid q < t \} \),
(ii) \( t \neq \sup \{ q \in \Lambda \mid q < t \} \).

Case (i) implies that
\[
x \in U(\alpha_A; t) \iff x \in I_q \quad \forall q < t \iff x \in \cap_{q < t} I_q,
\] (3.26)
so that \( U(\alpha_A; t) = \cap_{q < t} I_q \), which is an ideal of \( X \). For the case (ii), we claim that \( U(\alpha_A; t) = \cup_{q \geq t} I_q \). If \( x \in \cup_{q \geq t} I_q \), then \( x \in I_q \) for some \( q \geq t \). It follows that \( \alpha_A(x) \geq q \geq t \), so that \( x \in U(\alpha_A; t) \). This shows that \( \cup_{q \geq t} I_q \subseteq U(\alpha_A; t) \). Now assume that \( x \notin \cup_{q \geq t} I_q \). Then \( x \notin I_q \) for all \( q \geq t \). Since \( t \neq \sup \{ q \in \Lambda \mid q < t \} \), there exists \( \varepsilon > 0 \) such that \( (t - \varepsilon, t) \cap \Lambda = \emptyset \). Hence \( x \notin I_q \) for all \( q > t - \varepsilon \), which means that if \( x \in I_q \), then \( q < t - \varepsilon \). Thus \( \alpha_A(x) \leq t - \varepsilon < t \), and so \( x \notin U(\alpha_A; t) \). Therefore \( U(\alpha_A; t) \subseteq \cup_{q \geq t} I_q \), and thus \( U(\alpha_A; t) = \cup_{q \geq t} I_q \) which is an ideal of \( X \). Next we prove that \( L(\beta_A; s) \) is an ideal of \( X \). We consider the following two cases:

(iii) \( s = \inf \{ r \in \Lambda \mid s < r \} \),
(iv) \( s = \inf \{ r \in \Lambda \mid s \leq r \} \).

For the case (iii), we have
\[
x \in L(\beta_A; s) \iff x \in I_r \quad \forall s < r \iff x \in \cap_{s < r} I_r,
\] (3.27)
and hence \( L(\beta_A; s) = \cap_{s < r} I_r \), which is an ideal of \( X \). For the case (iv) there exists \( \varepsilon > 0 \) such that \( (s, s + \varepsilon) \cap \Lambda = \emptyset \). We will show that \( L(\beta_A; s) = \cup_{s \leq r} I_r \). If \( x \in \cup_{s \leq r} I_r \), then \( x \in I_r \) for some \( r \leq s \). It follows that \( \beta_A(x) \leq r \leq s \) so that \( x \in L(\beta_A; s) \). Hence \( \cup_{s \leq r} I_r \subseteq L(\beta_A; s) \). Conversely, if \( x \notin \cup_{s \leq r} I_r \), then \( x \notin I_r \) for all \( r \leq s \), which implies that \( x \notin I_r \) for all \( r < s + \varepsilon \), that is, if \( x \in I_r \), then \( r \geq s + \varepsilon \). Thus \( \beta_A(x) \geq s + \varepsilon > s \), that is, \( x \notin L(\beta_A; s) \). Therefore \( L(\beta_A; s) = \cup_{s \leq r} I_r \), and consequently \( L(\beta_A; s) = \cup_{s \leq r} I_r \) which is an ideal of \( X \). This completes the proof.

A mapping \( f : X \rightarrow Y \) of BCK-algebras is called a homomorphism if \( f(x * y) = f(x) * f(y) \) for all \( x, y \in X \). Note that if \( f : X \rightarrow Y \) is a homomorphism of BCK-algebras, then \( f(0) = 0 \). Let \( f : X \rightarrow Y \) be a homomorphism of BCK-algebras. For any IFSA = \( (\alpha_A, \beta_A) \) in \( Y \), we define a new IFSA\(^{f} = (\alpha_{A}^{f}, \beta_{A}^{f}) \) in \( X \) by
\[
\alpha_{A}^{f}(x) := \alpha_{A}(f(x)), \quad \beta_{A}^{f}(x) := \beta_{A}(f(x)) \quad \forall x \in X.
\] (3.28)

**Theorem 3.15.** Let \( f : X \rightarrow Y \) be a homomorphism of BCK-algebras. If an IFSA = \( (\alpha_A, \beta_A) \) in \( Y \) is an intuitionistic fuzzy ideal of \( Y \), then an IFSA\(^{f} = (\alpha_{A}^{f}, \beta_{A}^{f}) \) in \( X \) is an intuitionistic fuzzy ideal of \( X \).

**Proof.** We first have that
\[
\alpha_{A}^{f}(x) = \alpha_{A}(f(x)) \leq \alpha_{A}(0) = \alpha_{A}(f(0)) = \alpha_{A}^{f}(0),
\]
\[
\beta_{A}^{f}(x) = \beta_{A}(f(x)) \geq \beta_{A}(0) = \beta_{A}(f(0)) = \beta_{A}^{f}(0)
\] (3.29)
for all \( x \in X \). Let \( x, y \in X \). Then
\[
\begin{align*}
\min \{ \alpha^f_A(x \ast y), \alpha^f_A(y) \} &= \min \{ \alpha_A(f(x \ast y)), \alpha_A(f(y)) \} \\
&= \min \{ \alpha_A(f(x) \ast f(y)), \alpha_A(f(y)) \} \\
&\leq \alpha_A(f(x)) = \alpha^f_A(x), \\
\max \{ \beta^f_A(x \ast y), \beta^f_A(y) \} &= \max \{ \beta_A(f(x \ast y)), \beta_A(f(y)) \} \\
&= \max \{ \beta_A(f(x) \ast f(y)), \beta_A(f(y)) \} \\
&\geq \beta_A(f(x)) = \beta^f_A(x).
\end{align*}
\]

(3.30)

Hence \( A^f = (\alpha^f_A, \beta^f_A) \) is an intuitionistic fuzzy ideal of \( X \).

If we strengthen the condition of \( f \), then we can construct the converse of Theorem 3.15 as follows.

**Theorem 3.16.** Let \( f : X \to Y \) be an epimorphism of BCK-algebras and let \( A = (\alpha_A, \beta_A) \) be an IFS in \( Y \). If \( A^f = (\alpha^f_A, \beta^f_A) \) is an intuitionistic fuzzy ideal of \( X \), then \( A = (\alpha_A, \beta_A) \) is an intuitionistic fuzzy ideal of \( Y \).

**Proof.** For any \( x \in Y \), there exists \( a \in X \) such that \( f(a) = x \). Then

\[
\begin{align*}
\alpha_A(x) &= \alpha_A(f(a)) = \alpha^f_A(a) \leq \alpha^f_A(0) = \alpha_A(f(0)) = \alpha_A(0), \\
\beta_A(x) &= \beta_A(f(a)) = \beta^f_A(a) \geq \beta^f_A(0) = \beta_A(f(0)) = \beta_A(0).
\end{align*}
\]

(3.31)

Let \( x, y \in Y \). Then \( f(a) = x \) and \( f(b) = y \) for some \( a, b \in X \). It follows that

\[
\begin{align*}
\alpha_A(x) &= \alpha_A(f(a)) = \alpha^f_A(a) \\
&\geq \min \{ \alpha^f_A(a \ast b), \alpha^f_A(b) \} \\
&= \min \{ \alpha_A(f(a \ast b)), \alpha_A(f(b)) \} \\
&= \min \{ \alpha_A(f(a) \ast f(b)), \alpha_A(f(b)) \} \\
&= \min \{ \alpha_A(x \ast y), \alpha_A(y) \}, \\
\beta_A(x) &= \beta_A(f(a)) = \beta^f_A(a) \\
&\leq \max \{ \beta^f_A(a \ast b), \beta^f_A(b) \} \\
&= \max \{ \beta_A(f(a \ast b)), \beta_A(f(b)) \} \\
&= \max \{ \beta_A(f(a) \ast f(b)), \beta_A(f(b)) \} \\
&= \max \{ \beta_A(x \ast y), \beta_A(y) \}.
\end{align*}
\]

(3.32)

This completes the proof.

Let \( \text{IF}(X) \) be the family of all intuitionistic fuzzy ideals of \( X \) and let \( t \in [0, 1] \). Define binary relations \( U^t \) and \( L^t \) on \( \text{IF}(X) \) as follows:

\[
(A, B) \in U^t \iff U(\alpha_A; t) = U(\alpha_B; t), \quad (A, B) \in L^t \iff L(\beta_A; t) = L(\beta_B; t),
\]

(3.33)

respectively, for \( A = (\alpha_A, \beta_A) \) and \( B = (\alpha_B, \beta_B) \) in \( \text{IF}(X) \). Then clearly \( U^t \) and \( L^t \) are
equivalence relations on $\text{IF}(X)$. For any $A = (\alpha_A, \beta_A) \in \text{IF}(X)$, let $[A]_{U^t}$ (respectively, $[A]_{L^t}$) denote the equivalence class of $A$ modulo $U^t$ (respectively, $L^t$), and denote by $\text{IF}(X)/U^t$ (respectively, $\text{IF}(X)/L^t$) the system of all equivalence classes modulo $U^t$ (respectively, $L^t$); so

$$\text{IF}(X)/U^t := \{[A]_{U^t} | A = (\alpha_A, \beta_A) \in \text{IF}(X)\},$$  \hfill (3.34)

respectively,

$$\text{IF}(X)/L^t := \{[A]_{L^t} | A = (\alpha_A, \beta_A) \in \text{IF}(X)\}.$$  \hfill (3.35)

Now let $I(X)$ denote the family of all ideals of $X$ and let $t \in [0, 1]$. Define maps $f_t$ and $g_t$ from $\text{IF}(X)$ to $I(X) \cup \{\emptyset\}$ by $f_t(A) = U(\alpha_A; t)$ and $g_t(A) = L(\beta_A; t)$, respectively, for all $A = (\alpha_A, \beta_A) \in \text{IF}(X)$. Then $f_t$ and $g_t$ are clearly well defined.

**Theorem 3.17.** For any $t \in (0, 1)$ the maps $f_t$ and $g_t$ are surjective from $\text{IF}(X)$ to $I(X) \cup \{\emptyset\}$.

**Proof.** Let $t \in (0, 1)$. Note that $0_- = (0, 1)$ is in $\text{IF}(X)$, where $0$ and $1$ are fuzzy sets in $X$ defined by $0(x) = 0$ and $1(x) = 1$ for all $x \in X$. Obviously $f_t(0_-) = U(0; t) = \emptyset = L(1; t) = g_t(0_-)$. Let $G(\neq \emptyset) \in I(X)$. For $G_- = (\chi_G, \bar{\chi}_G) \in \text{IF}(X)$, we have $f_t(G_-) = U(\chi_G; t) = G$ and $g_t(G_-) = L(\bar{\chi}_G; t) = G$. Hence $f_t$ and $g_t$ are surjective. \hfill $\Box$

**Theorem 3.18.** The quotient sets $\text{IF}(X)/U^t$ and $\text{IF}(X)/L^t$ are equipotent to $I(X) \cup \{\emptyset\}$ for every $t \in (0, 1)$.

**Proof.** For $t \in (0, 1)$ let $f_t^*$ (respectively, $g_t^*$) be a map from $\text{IF}(X)/U^t$ (respectively, $\text{IF}(X)/L^t$) to $I(X) \cup \{\emptyset\}$ defined by $f_t^*([A]_{U^t}) = f_t(A)$ (respectively, $g_t^*([A]_{L^t}) = g_t(A)$) for all $A = (\alpha_A, \beta_A) \in \text{IF}(X)$. If $U(\alpha_A; t) = U(\alpha_B; t)$ and $L(\beta_A; t) = L(\beta_B; t)$ for $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ in $\text{IF}(X)$, then $(A, B) \in U^t$ and $(A, B) \in L^t$; hence $[A]_{U^t} = [B]_{U^t}$ and $[A]_{L^t} = [B]_{L^t}$. Therefore the maps $f_t^*$ and $g_t^*$ are injective. Now let $G(\neq \emptyset) \in I(X)$. For $G_- = (\chi_G, \bar{\chi}_G) \in \text{IF}(X)$, we have

$$f_t^*([G_-]_{U^t}) = f_t(G_-) = U(\chi_G; t) = G,$$

$$g_t^*([G_-]_{L^t}) = g_t(G_-) = L(\bar{\chi}_G; t) = G.$$  \hfill (3.36)

Finally, for $0_- = (0, 1) \in \text{IF}(X)$ we get

$$f_t^*([0_-]_{U^t}) = f_t(0_-) = U(0; t) = \emptyset,$$

$$g_t^*([0_-]_{L^t}) = g_t(0_-) = L(0; t) = \emptyset.$$  \hfill (3.37)

This shows that $f_t^*$ and $g_t^*$ are surjective. This completes the proof. \hfill $\Box$

For any $t \in [0, 1]$, we define another relation $R^t$ on $\text{IF}(X)$ as follows:

$$(A, B) \in R^t \iff U(\alpha_A; t) \cap L(\beta_A; t) = U(\alpha_B; t) \cap L(\beta_B; t)$$  \hfill (3.38)
for any $A = (\alpha_A, \beta_A), B = (\alpha_B, \beta_B) \in \text{IF}(X)$. Then the relation $R^t$ is also an equivalence relation on $\text{IF}(X)$.

**Theorem 3.19.** For any $t \in (0, 1)$, the map $\phi_t : \text{IF}(X) \to I(X) \cup \{\emptyset\}$ defined by $\phi_t(A) = f_t(A) \cap g_t(A)$ for each $A = (\alpha_A, \beta_A) \in \text{IF}(X)$ is surjective.

**Proof.** Let $t \in (0, 1)$. For $0_\ast = (0, 1) \in \text{IF}(X)$,

$$\phi_t(0_\ast) = f_t(0_\ast) \cap g_t(0_\ast) = U(0; t) \cap L(1; t) = \emptyset.$$ 

(3.39)

For any $H \in \text{IF}(X)$, there exists $H_\ast = (\chi_H, \bar{\chi}_H) \in \text{IF}(X)$ such that

$$\phi_t(H_\ast) = f_t(H_\ast) \cap g_t(H_\ast) = U(\chi_H; t) \cap L(\bar{\chi}_H; t) = H.$$ 

(3.40)

This completes the proof.

**Theorem 3.20.** For any $t \in (0, 1)$, the quotient set $\text{IF}(X)/R^t$ is equipotent to $I(X) \cup \{\emptyset\}$.

**Proof.** Let $t \in (0, 1)$ and let $\phi_t^* : \text{IF}(X)/R^t \to I(X) \cup \{\emptyset\}$ be a map defined by $\phi_t^*([A]_{R^t}) = \phi_t(A)$ for all $[A]_{R^t} \in \text{IF}(X)/R^t$. If $\phi_t^*([A]_{R^t}) = \phi_t^*([B]_{R^t})$ for any $[A]_{R^t}, [B]_{R^t} \in \text{IF}(X)/R^t$, then $f_t(A) \cap g_t(A) = f_t(B) \cap g_t(B)$, that is, $U(\alpha_A; t) \cap L(\beta_A; t) = U(\alpha_B; t) \cap L(\beta_B; t)$, hence $(A, B) \in R^t$. It follows that $[A]_{R^t} = [B]_{R^t}$ so that $\phi_t^*$ is injective. For $0_\ast = (0, 1) \in \text{IF}(X)$,

$$\phi_t^*([0_\ast]_{R^t}) = \phi_t(0_\ast) = f_t(0_\ast) \cap g_t(0_\ast) = U(0; t) \cap L(1; t) = \emptyset.$$ 

(3.41)

If $H \in \text{IF}(X)$, then for $H_\ast = (\chi_H, \bar{\chi}_H) \in \text{IF}(X)$, we have

$$\phi_t^*([H_\ast]_{R^t}) = \phi(H_\ast) = f_t(H_\ast) \cap g_t(H_\ast) = U(\chi_H; t) \cap L(\bar{\chi}_H; t) = H.$$ 

(3.42)

Hence $\phi_t^*$ is surjective, this completes the proof.

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