DERIVATIONS OF CERTAIN OPERATOR ALGEBRAS

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Abstract. Let $\mathcal{N}$ be a nest and let $\mathcal{A}$ be a subalgebra of $L(H)$ containing all rank one operators of $\mathcal{N}$. We give several conditions under which any derivation $\delta$ from $\mathcal{A}$ into $L(H)$ must be inner. The conditions include (1) $H_+ \neq H$, (2) $0_+ \neq 0$, (3) there is a nontrivial projection in $\mathcal{N}$ which is in $\mathcal{A}$, and (4) $\delta$ is norm continuous. We also give some applications.

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1. Introduction. In this paper, we unify some results on derivations by considering derivations from an algebra $\mathcal{A}$ containing all rank one operators of a nest algebra into an $\mathcal{A}$-bimodule $\mathcal{B}$. Chernoff [1] proves that every derivation from $F(H)$ into $L(H)$ is inner. In [2], Christensen proves that every derivation from a nest algebra into itself or into $L(H)$ is inner. In [3], Christensen and Peligrad show that every derivation of a quasitriangular operator algebra into itself is inner. Knowles [7] generalizes the result of [2] and gets that any derivation from a nest algebra into an ideal $\mathcal{J}$ of $L(H)$ is inner.

Let $\mathcal{N}$ be a nest of subspaces of a Hilbert space $H$, let $\mathcal{A}$ be a subalgebra of $L(H)$ containing all rank one operators of $\mathcal{N}$, and let $\delta$ be a derivation from $\mathcal{A}$ into $L(H)$. We prove that if one of the following conditions holds:

1. $H_+ \neq H$,
2. $0_+ \neq 0$,
3. there exists a nontrivial $P \in \mathcal{N}$, such that $P \in \mathcal{A}$, then $\delta$ is inner.

We also prove that for any nest, if $\delta$ is a norm continuous derivation from $\mathcal{A}$ into $L(H)$, then $\delta$ is inner.

We discuss some applications of these results.

Let $H$ be a complex separable Hilbert space, $L(H)$ the algebra of all bounded linear operators on $H$, $K(H)$ the ideal of all compact operators in $L(H)$, $F(H)$ the subalgebra of all finite rank operators on $H$, and $F_1(H)$ the subset of all operators in $F(H)$ with rank less than or equal to 1. We call a subalgebra $\mathcal{A}$ of $L(H)$ standard provided $\mathcal{A}$ contains $F(H)$. A collection $\mathcal{E}$ of subspaces of $H$ is said to be a subspace lattice if it contains zero and $H$, and is complete in the sense that it is closed under the formation of arbitrary closed linear spans and intersections. A subspace lattice $\mathcal{N}$ is called a nest if it is a totally ordered subspace lattice. Given a nest $\mathcal{N}$, let $\text{alg}\mathcal{N} = \{ T \in L(H) : TN \subseteq N, N \in \mathcal{N} \}$. Alg$\mathcal{N}$ is said to be the nest algebra associated with $\mathcal{N}$. If $\mathcal{N}$ is a nest and $E \in \mathcal{N}$, then we define $E_- = \lor \{ F \in \mathcal{N} : F \subseteq E \}$, and $E_+ = \land \{ F \in \mathcal{N} : F \supseteq E \}$. If $e, f \in H$ we write $e^* \otimes f$ for the rank one operator $x \rightarrow (x, e)f$, whose norm is $\|e\|\|f\|$. If $\mathcal{N}$ is a nest, then by [8, Lemma 3.7], $e^* \otimes f \in \text{alg}\mathcal{N}$ if and only if there is an $E \in \mathcal{N}$...
such that $f \in E$ and $e \in (E\Lambda)^{\perp}$. If $\mathcal{A}$ is a subalgebra of $L(H)$, then we say that $\mathcal{A}$ is a triangular operator algebra, if $\mathcal{A} \cap \mathcal{A}^{*}$ is a maximal abelian selfadjoint subalgebra of $L(H)$. If $\mathcal{A}$ is maximal triangular, and let $\mathcal{A}$ is a maximal nest, then we say that $\mathcal{A}$ is strongly reducible. A derivation $\delta$ of an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{B}$ is a linear map satisfying $\delta(AB) = A\delta(B) + \delta(A)B$, for any $A, B \in \mathcal{A}$. A derivation $\delta$ is called $\mathcal{B}$-inner if there exists $T \in \mathcal{B}$, such that $\delta(A) = AT - TA$. When we say that a derivation $\delta : \mathcal{A} \to \mathcal{B}$ is inner, we mean $\mathcal{B}$-inner.

2. Derivations

Let $N$ be a nest. In the following, we consider the derivation from a subalgebra $\mathcal{A}$ of $L(H)$ containing all rank one operators of $\text{alg}N$ into $L(H)$.

**Theorem 2.1.** If $N$ is a nest such that $H_{N} \neq H$, $\mathcal{A}$ is a subalgebra of $L(H)$ containing $(\text{alg}N) \cap F_{1}(H)$, and $\delta$ is a derivation from $\mathcal{A}$ into $L(H)$, then $\delta$ is inner.

**Proof.** Since $H_{N} \neq H$, for any $f^{*} \in (H_{N})^{\perp}$, $f^{*} \neq 0$, we choose $\gamma$ in $H$ such that $f^{*}(\gamma) = 1$. For any $x$ in $H$, by [8, Lemma 3.7], it follows that $f^{*}\otimes x \in \text{alg}N$. Now define

$$Tx = -\delta(f^{*}\otimes x)\gamma, \quad \text{for } x \in H.$$  \hspace{1cm} (2.1)

Now for $A$ in $\mathcal{A}$,

$$TAx = -\delta(f^{*}\otimes Ax)\gamma = -\delta(A)x - A\delta(f^{*}\otimes x)\gamma = -\delta(A)x + ATx.$$  \hspace{1cm} (2.2)

Hence for any $x \in H$, $-TAx + ATx = \delta(A)x$; thus

$$\delta(A) = AT - TA.$$  \hspace{1cm} (2.3)

It remains to show that $\delta$ is bounded.

Let $\lim_{n \to \infty}X_{n} = x$, and $\lim_{n \to \infty}TX_{n} = y$. Now for any rank one operator $A \in \text{alg}N$, we have that $\delta(A)$ and $TA$ are bounded. It follows that $AT = \delta(A) + TA$ is bounded, and $\lim_{n \to \infty}ATX_{n} = ATx = Ay$. Since $\mathcal{A}$ contains all rank one operators of $\text{alg}N$, and by [4, Proposition 3.8], every finite rank operator of $\text{alg}N$ is a sum of some rank one operators of $\text{alg}N$, we have, for any finite rank operator $B$ in $\text{alg}N$, $BTx = By$. By [4, Theorem 3.11], choose a bounded net $\{B_{\lambda}\}$ of finite rank operators in $\text{alg}N$ such that $\lim_{\lambda}B_{\lambda} = I$ in the strong operator topology. We have $TX = y$. By the closed graph theorem, it follows that $T$ is bounded. \hfill \square

**Corollary 2.2.** If $N$ is a nest such that $0_{\perp} \neq 0$, and $\mathcal{A}$ is a subalgebra of $L(H)$ containing all rank one operators of $\text{alg}N$, then every derivation $\delta$ from $\mathcal{A}$ into $L(H)$ is inner.

**Proof.** Let $\mathcal{N}^{\perp} = \{N_{\perp} : N \in N\}$. Then $\mathcal{N}^{\perp}$ is a nest such that $H_{\mathcal{N}^{\perp}} \neq H$. Since $\text{alg}\mathcal{N}^{\perp} = (\text{alg}N)^{*}$, it follows that $\mathcal{A}^{*}$ contains all rank one operators of $\text{alg}\mathcal{N}^{\perp}$. Define $\delta^{*}(A = (\delta(A^{*})))^{*}$ for any $A$ in $\mathcal{A}^{*}$.

It is easy to prove that $\delta^{*}$ is a derivation from $\mathcal{A}^{*}$ into $L(H)$. By Theorem 2.1, we have that $\delta^{*}$ is inner. It follows that $\delta$ is inner. \hfill \square

We now consider a nest $N$ such that $H_{N} = H$. 


**Lemma 2.3.** Let $\mathcal{N}$ be a nest, $E_1,E_2 \in \mathcal{N}$ and $E_1 \subset E_2$. If $T$ is a linear map from $E_2$ into $H$ such that $S = TS$ on $E_2$ for any rank one operator $S$ of $\text{alg}\mathcal{N}$, then there exists a $\lambda$ such that $Tx = \lambda x$, for any $x \in E_1$.

**Proof.** For $x \in E_1$, choose $y \in E_2 - E_1$ such that $\|y\| = 1$. Since $y^* \otimes x \in \text{alg}\mathcal{N}$, by hypothesis

\[ Ty^* \otimes x(y) = y^* \otimes xTy = Tx = (Ty,y)x. \tag{2.4} \]

Since every one-dimensional subspace of $L(E_2,H)$ is reflexive, it follows that there exists $\lambda$ such that $T = \lambda I$. \hfill $\Box$

**Lemma 2.4.** Let $\mathcal{N}$ be a nest such that $H_- = H$, and let $M = \cup \{ N : N \subsetneq H, N \in \mathcal{N} \}$. Then there exists a linear map $T$ from $M$ into $H$ such that $\delta(A)x = (AT - TA)x$, for any $x \in M$.

**Proof.** Since $H_- = H$, we may choose an increasing sequence $\{ P_i \} \subset \mathcal{N}$ such that $P_i \to I$ in the strong operator topology. Also choose $f^* \in P_i^\perp$, and $y \in H$, such that $\|f^*\| = 1$, $f^*(y) = 1$, and $\|y\| \leq 2$. Define,

\[ T_i x = -\delta(f^* \otimes x)y \quad \text{for} \quad x \in P_i. \tag{2.5} \]

Using an argument similar to the proof of Theorem 2.1, we may prove that for $A$ in $\mathcal{A}$, $\delta(A)x = (AT_i - T_iA)x$ for $x$ in $P_i$. If $j \geq i$, then for $x \in P_i$, $(AT_i - T_iA)x = (AT_j - T_jA)x$. Hence

\[ A(T_i - T_j)x = (T_i - T_j)Ax, \quad \text{for} \quad x \in P_i. \tag{2.6} \]

By Lemma 2.3, we have $T_j - T_i = \lambda_{ij}$ on $P_{i-1}$. Now for $j > i$, let $\tilde{T}_j = T_i + \lambda_{1,j}$. We have, for $k > j > 2$, $\tilde{T}_j x = \tilde{T}_k x$ for all $x \in P_{j-1}$. Now for any $x \in \cup \{ P_i \} = \cup \{ N : N \subsetneq H, N \in \mathcal{N} \}$, choose a $j > 2$ such that $x \in P_j$ and let $Tx = \tilde{T}_j x$. Then, $T$ is well defined and for $x$ in $M$, $\delta(A)x = (AT - TA)x$. \hfill $\Box$

**Remark 2.5.** Using the idea in the proof of Theorem 2.1, we can prove that in Lemma 2.3, $T_i$ is a bounded operator from $P_i$ into $H$.

**Theorem 2.6.** If $\mathcal{N}$ is a nest, $\mathcal{A}$ is a subalgebra of $L(H)$ containing all rank one operators of $\text{alg}\mathcal{N}$, and $\delta$ is a norm continuous derivation from $\mathcal{A}$ into $L(H)$, then $\delta$ is inner.

**Proof.** If $\mathcal{N}$ satisfies $H_- \neq H$, then by Theorem 2.1, we get that $\delta$ is inner. If $\mathcal{N}$ satisfies $H_- = H$, then by Lemma 2.4, there exists a linear map $T$ such that

\[ \delta(A)x = (AT - TA)x \quad \text{for any} \quad x \in M = \cup \{ N : N \subsetneq H, N \in \mathcal{N} \}. \tag{2.7} \]

By (2.5) and the boundedness of $\delta$, it follows that $\|T_i x\| \leq 2\|\delta\| \|x\|$. Since $|\lambda_{ij}| \leq \|T_i\| + \|T_j\| \leq 4\|\delta\|$, it follows that $\|T\| \leq 6\|\delta\|$. Thus $T$ is bounded on $M$. Let $\tilde{T}$ be the unique bounded extension of $T$ to $H$. Then $\tilde{T}$ is bounded and for $A$ in $\mathcal{A}$, $\delta(A) = A\tilde{T} - \tilde{T}A$. \hfill $\Box$
**Theorem 2.7.** Let \( \mathcal{N} \) be a nest satisfying \( H_1 = H \). If there exists a nontrivial projection \( P \in \mathcal{N} \), such that \( P \in \mathcal{A} \), and \( \delta \) is a derivation from \( \mathcal{A} \) into \( L(H) \), then \( \delta \) is inner.

**Proof.** As in the proof of Lemma 2.4, we choose \( P_1 = P \). Let \( H = P \oplus P^\perp \). Then \( T \) can be decomposed as

\[
T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.
\]

Let \( Q = \cup \{ N - P : P \subseteq N \in \mathcal{N}, N \neq H \} \), \( T_{12} : Q \rightarrow P \), \( T_{22} : Q \rightarrow Q \).

By the definition of \( T \), \( T_{11} \) and \( T_{21} \) are bounded. We now prove that \( T_{12} \) and \( T_{22} \) are bounded. Since \( A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) in \( \mathcal{A} \), we have that \( \delta(A) = \begin{pmatrix} 0 & T_{12} \\ -T_{21} & 0 \end{pmatrix} \) holds on \( M \). Since \( \delta(A) \) is bounded, it follows that \( T_{12} \) is bounded. Now, for any rank one operator \( A \in L(H) \), we have \( PA(1 - P) \in \mathcal{A} \). Hence,

\[
\delta(PA(1 - P)) = \begin{pmatrix} PA(1 - P) & PA(1 - P)T_{22} - T_{11} \\ 0 & -T_{21}PA(1 - P) \end{pmatrix}
\]

(2.9)

holds on \( M \). Since \( \delta(PA(1 - P)) \) is bounded, it follows that \( PA(1 - P)T_{22} \) is bounded. Hence for any \( f^* \in P^\perp \) and \( e \in P, e \neq 0 \), \( f^* \otimes eT_{22} \) is bounded on \( Q \). Thus there exists \( c \) such that \( |f^*(T_{22}x)| \leq c \), for any \( x \in Q \), and \( \|x\| \leq 1 \). By the uniform boundedness theorem, we have that \( \{ \|T_{22}x\| : \|x\| \leq 1 \} \) is bounded. Hence \( T_{22} \) is bounded. As in Theorem 2.6, there exists a bounded extension \( \tilde{T} \) of \( T \) to \( H \) such that for \( A \) in \( \mathcal{A} \), \( \delta(A) = A\tilde{T} - \tilde{T}A \).

\[ \square \]

3. Applications. In this section, we apply the results above to some special subalgebras of \( L(H) \). If \( A \supseteq F(H) \), then by Theorem 2.1, we have the following corollaries.

**Corollary 3.1** [1]. Every derivation from a standard operator algebra into \( L(H) \) is inner.

**Corollary 3.2** [2]. If \( \delta \) is a derivation from \( \text{alg} \mathcal{N} \) into itself, then \( \delta \) is inner.

**Proof.** By Theorems 2.1 and 2.7, we have that there is \( T \) in \( L(H) \) such that for any \( A \) in \( \mathcal{A} \), \( \delta(A) = A\tilde{T} - \tilde{T}A \). Now we prove that \( T \) is in \( \text{alg} \mathcal{N} \). Now for any \( P \) in \( \mathcal{N} \), since \( \delta(P) = PT - TP \) in \( \text{alg} \mathcal{N} \), we have that \( (I - P)\delta(P)P = 0 = -(I - P)TP \). This completes the proof. \[ \square \]

Let \( \mathcal{B} \) be a subalgebra of \( L(H) \), and let \( \mathcal{I} \) be any subset of \( L(H) \). We denote by \( C(\mathcal{B}, \mathcal{I}) \) the collection, \( \{ T \in L(H) : AT - TA \in \mathcal{I} \}, \forall A \in \mathcal{B} \} \).

**Lemma 3.3** [6]. Let \( \mathcal{B} \) be a nest algebra and \( \mathcal{I} \) be an ideal in \( L(H) \). Then \( C(\mathcal{B}, \mathcal{I}) = CI + \mathcal{I} \).

Using this lemma and Theorem 2.7, we easily prove the following result.

**Corollary 3.4.** If \( \mathcal{B} \) is an algebra containing \( \text{alg} \mathcal{N} \), then any derivation \( \delta : \mathcal{B} \rightarrow C_p \) is inner for \( 1 \leq p \leq \infty \).

**Corollary 3.5.** If \( \mathcal{B} \) is a triangular operator algebra containing every rank one operator in \( \text{alg} \mathcal{N} \), then every derivation \( \delta \) from \( \mathcal{B} \) into \( L(H) \) is inner.
**Proof.** Suppose $\tilde{N}$ is a maximal nest containing $N$. By hypothesis we have that $B \supseteq (\text{alg } N) \cap F_1(H) \supseteq (\text{alg } \tilde{N}) \cap F_1(H)$. Since $B$ contains all rank one operators of alg $N$, we have that lat $B \subseteq N$. By [5, Theorem 4], it follows that lat $B = \tilde{N} = N$. Since $B$ is a triangular operator algebra, it follows $\tilde{N} \subseteq B$.

If $H_+ \neq H$, then by Theorem 2.1, we have that $\delta$ is inner.

If $H_+ = H$, $N \subseteq B$, and $N$ is a maximal nest, by Theorem 2.7, it follows that $\delta$ is inner.

**Remark 3.6.** By Corollary 3.1, it follows that every derivation $\delta : F(H) \to L(H)$ is inner. However if $B$ is a unital algebra containing $F(H)$ and $B \neq L(H)$, then there is a derivation from $F(H)$ into $B$ that is not inner, e.g., $\delta = \delta_T$ with $T \notin B$. Also if $\mathcal{A} = K(H) + CI$, and $T \notin \mathcal{A}$, then $\delta_T : \mathcal{A} \to \mathcal{A}$ is a derivation that is not inner, but $\mathcal{A}$ contains all rank one operators of $L(H)$.

By [8, Lemma 5.2], we know that if $B$ is a strongly reducible maximal triangular algebra, then lat $B$ is a nest and $B$ contains all rank one operators of alg lat $(B)$. Hence by Corollary 3.5 and Theorem 2.7, we have the following result.

**Corollary 3.7.** Every derivation from a strongly reducible maximal triangular algebra into $L(H)$ is inner.

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