SUBORDINATION BY CONVEX FUNCTIONS

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(Received 18 January 1990)

Abstract. Let \( K(\alpha), 0 \leq \alpha < 1 \), denote the class of functions \( g(z) = z + \sum_{n=2}^{\infty} a_n z^n \) which are regular and univalently convex of order \( \alpha \) in the unit disc \( U \). Pursuing the problem initiated by Robinson in the present paper, among other things, we prove that if \( f \) is regular in \( U \), \( f(0) = 0 \), and \( f(z) + z f'(z) < g(z) + z g'(z) \) in \( U \), then (i) \( f(z) < g(z) \) at least in \( |z| < r_0, r_0 = \sqrt{5/3} = 0.745... \) if \( f \in K \); and (ii) \( f(z) < g(z) \) at least in \( |z| < r_1, r_1 = (51 - 24\sqrt{2})/23)^{1/2} = 0.8612... \) if \( g \in K(1/2) \).

Keywords and phrases. Subordination, convex function, convex function of order 1/2.

2000 Mathematics Subject Classification. Primary 30C45.

1. Introduction. Let \( S \) denote the class of functions \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) that are regular and univalent in the unit disc \( U = \{z/|z| < 1\} \). For a given \( \alpha, 0 \leq \alpha < 1 \), denote by \( K(\alpha) \) the subclass of \( S \) consisting of functions \( f \) which satisfy the condition

\[
\Re \left( Z - \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in U.
\]

(1.1)

\( K(\alpha) \) is called the class of convex functions of order \( \alpha \) and \( K = K(0) \) is the class of convex functions.

Suppose that \( f \) and \( g \) are regular in \( |z| < \rho \) and \( f(0) = g(0) \). In addition, suppose that \( g \) is also univalent in \( |z| < \rho \). We say that \( f \) is subordinate to \( g \) in \( |z| < \rho \) (in symbols, \( f(z) < g(z) \) in \( |z| < \rho \) if \( f(|z| < \rho) \subset g(|z| < \rho) \).

In 1947, Robinson [2] proved that if \( g(z) + zg'(z) \) is in \( S \) and \( f(z) + zf'(z) < g(z) + zg'(z) \) in \( |z| < 1 \), then \( f(z) < g(z) \) at least in \( |z| < r_0 = 1/5 \). Subsequently, Singh and Singh [4] increased the constant \( r_0 \) to \( 2 - \sqrt{3} = 0.268... \) Miller, Mocanu, and Read [1] further increased the constant to \( 4 - \sqrt{13} = 0.3944... \).

Here, we consider the problem of Robinson when \( g \in K \) and \( K(1/2) \), respectively. (It is easy to see that \( g(z) + zg'(z) \) is close-to-convex and hence univalent in \( |z| < 1 \) when \( g \in K \).) We remark that our method works even when \( g \in K(\alpha) \). However, calculations in this general case become so cumbersome that the result obtained does not commensurate with the input labour. We, therefore, confine ourselves to the particular cases \( \alpha = 0 \) and \( \alpha = 1/2 \).

2. Preliminaries. We need the following results.

\textbf{Lemma 2.1.} Suppose that \( f \) and \( g \) are regular in \( U \), \( f(0) = g(0) \), and \( g'(0) \neq 0 \). Suppose further that

\[
\Re \left( 1 + \frac{zf''(z)}{g'(z)} \right) > -\frac{1}{2}, \quad z \in U.
\]

(2.1)
Then if \( f(z) \prec g(z) \) in \( U \), we have
\[
\frac{1}{z} \int_0^z f(t) \, dt < \frac{1}{z} \int_0^z g(t) \, dt, \quad z \in U.
\] (2.2)

We observe that (2.1) implies that \( g \) is close-to-convex and hence univalent in \( U \) and that the right-hand side function in (2.2) is convex in \( U \) [3]. Lemma 2.1 is due to Miller, Mocanu, and Reade [1].

The underlying idea of the following result is essentially due to Zomorvić [6] (also, see [5]).

**Lemma 2.2.** Let \( P \) be regular in \( U \), \( P(0) = 1 \), and \( \text{Re} P(z) > 0 \) in \( U \). Let \( \mu \) and \( \lambda \) be fixed real numbers, \(-\infty < \mu < \infty, \lambda \geq 0, \) and \( |z| = r < 1 \). Then
\[
\text{Re} \left[ \mu P(z) + \frac{zP'(z)}{P(z) + \lambda} \right] \geq \begin{cases} 
- \frac{(\sqrt{\lambda(\mu+1)} - \sqrt{(a+\lambda)})^2}{2}, & \text{if } \frac{\lambda(a+\lambda)}{(a-\rho+\lambda)^2} \geq \mu + 1 \\
(a-\rho) \left( \mu - \frac{\rho}{a-\rho+\lambda} \right), & \text{if } \mu + 1 > \frac{\lambda(a+\lambda)}{(a-\rho+\lambda)^2}, \\
(a+\rho) \left( \mu + \frac{\rho}{a+\rho+\lambda} \right), & \text{if } \mu + 1 < \frac{\lambda(a+\lambda)}{(a+\rho+\lambda)^2},
\end{cases}
\] (2.3)

where \( a = (1+r^2)/(1-r^2) \) and \( \rho = 2r/(1-r^2) \).

**Proof.** Making use of the inequality (2.3) (see [5])
\[
\left| zP'(z) - \frac{P^2(z) - 1}{2} \right| \leq \frac{\rho^2 - \rho_0^2}{2},
\] (2.4)
where \( |P(z) - a| = \rho_0 \leq \rho \), we get
\[
\text{Re} \left[ \mu P(z) + \frac{zP'(z)}{P(z) + \lambda} \right] \geq \text{Re} \left[ \mu P(z) + \frac{P(z) - \lambda}{2} + \frac{\lambda^2 - 1}{2|P(z) + \lambda|^2} \right] - \frac{\rho^2 - \rho_0^2}{2|P(z) + \lambda|^2}.
\] (2.5)

Taking \( P(z) = a + \xi + i\eta \) and \( R_1^2 = (a+\xi+\lambda)^2 + \eta^2 \), we get
\[
\text{Re} \left[ \mu P(z) + \frac{zP'(z)}{P(z) + \lambda} \right] \geq \mu(a + \xi) + \frac{\lambda^2 - 1}{2R_1^2} - \frac{\rho^2 - \xi^2 - \eta^2}{2R_1} = S(\xi, \eta).
\] (2.6)

Now it is easy to see that \( \partial S(\xi, \eta)/\partial \eta = 0 \) and \( \partial^2 S(\xi, \eta)/\partial \eta^2 > 0 \) at \( \eta = 0 \). Therefore,
\[
\min_\eta S(\xi, \eta) = S(\xi, 0) = \mu(a + \xi) + \frac{\lambda^2 - 1}{2(a+\xi+\lambda)} - \frac{\rho^2 - \xi^2}{2(a+\xi+\lambda)} = \frac{(\mu + 1)R + \lambda(a+\lambda)}{R} - (\mu + 2)\lambda - a = L(R),
\] (2.7)
where \( R = a + \xi + \lambda \). Now, using the fact that \( |R(z) - a| < \rho \), we obtain the inequality
\[
a - \rho + \lambda \leq R \leq a + \rho + \lambda. \tag{2.8}
\]

It is observed that at \( R = R_0 = \frac{(\lambda(a + \lambda)/(\mu + 1))^{1/2}}{\partial L(R)/\partial R = 0} \) and \( \partial^2 L(R)/\partial R^2 > 0 \). Thus, \( R = R_0 \) gives the minimum value of \( L(R) \) provided \( R_0 \) lies in the range of \( R \).

In view of (2.8), this is the case if the inequality
\[
\frac{\lambda(a + \lambda)}{(a - \rho + \lambda)^2} \geq \mu + 1 \geq \frac{\lambda(a + \lambda)}{(a + \rho + \lambda)^2} \tag{2.9}
\]
is satisfied. Thus, if (2.9) holds, we have
\[
\min_R L(R) = L(R_0) = -\left(\sqrt{\lambda(\mu + 1)} - \sqrt{\lambda + a}\right)^2. \tag{2.10}
\]

Also, it is easy to check that when \( \mu + 1 > \lambda(a + \lambda)/(a - \rho + \lambda)^2 \), \( L(R) \) is an increasing function of \( R \). Therefore, in this case,
\[
\min_R L(R) = L(a - \rho + \lambda) = (a - \rho)\left(\mu - \frac{\rho}{a - \rho + \lambda}\right). \tag{2.11}
\]

On the other hand, when \( \mu + 1 < \lambda(a + \lambda)/(a + \rho + \lambda)^2 \), \( L(R) \) is a decreasing function of \( R \). Therefore, in this case,
\[
\min_R L(R) = L(a + \rho + \lambda) = (a + \rho)\left(\mu + \frac{\rho}{a + \rho + \lambda}\right). \tag{2.12}
\]

This completes the proof of Lemma 2.2.

\( \square \)

3. Theorems and their proofs

**Theorem 3.1.** Let \( f \) be regular in \( U \) with \( f(0) = 0 \) and let \( g \in K \). Suppose that
\[
f(z) + zf'(z) \prec g(z) + zg'(z) \quad \text{in} \ U. \tag{3.1}
\]

Then \( f(z) < g(z) \) at least in \( |z| < r_0 \), where \( r_0 = \sqrt{5}/3 = 0.745 \ldots. \)

**Proof.** Let us take
\[
h(z) = g(z) + zg'(z). \tag{3.2}
\]

Since \( g \in K \), we can put
\[
1 + \frac{zg''(z)}{g'(z)} = P(z), \tag{3.3}
\]
where \( P(z) \) is regular in \( U \), \( P(0) = 1 \), and \( \text{Re} P(z) > 0 \) in \( U \). Now, from (3.2) and (3.3), we get
\[
1 + \frac{zh''(z)}{h'(z)} = P(z) + \frac{zP'(z)}{P(z) + 1}. \tag{3.4}
\]
Taking $\mu = \lambda = 1$ in Lemma 2.2, we easily obtain

$$\text{Re} \left( 1 + \frac{zh''(z)}{h'(z)} \right) \geq \begin{cases} \frac{1 - 2r}{1 + r}, & \text{if } 0 \leq r < \frac{3}{5}, \\ -2 \left[ 1 - \frac{a}{\sqrt{1 - r^2}} \right]^2, & \text{if } \frac{3}{5} \leq r < 1, \end{cases}$$

(3.5)

where $|z| = r < 1$. Now, it is easy to verify that for $0 \leq r < 3/5$, $\text{Re}(1 + zh''(z)/h'(z)) > -1/2$ and for $3/5 \leq r < 1$, $\text{Re}(1 + zh''(z)/h'(z)) > -1/2$ whenever $9r^4 + 22r^2 - 15 < 0$ or whenever $r < r_0$, where $r_0 = \sqrt{5}/3$ is the smallest positive root of $9r^4 + 22r^2 - 15 = 0$. The assertion of our theorem now follows from Lemma 2.1.

**Theorem 3.2.** Let $f$ be regular in $U$ with $f(0) = 0$ and let $g \in K(1/2)$. Suppose that

$$f(z) + zf'(z) < g(z) + zg'(z) \quad \text{in } U.$$  

(3.6)

Then

$$f(z) < g(z)$$

(3.7)

at least in $|z| < r_1$, where $r_1 = ((51 - 24\sqrt{2})/23)^{1/2} = 0.8612\ldots$.

**Proof.** Let us put

$$h(z) = g(z) + zg'(z).$$

(3.8)

Since $g \in K(1/2)$, we can write

$$1 + \frac{zg''(z)}{g'(z)} = \frac{P(z) + 1}{2},$$

(3.9)

where $P(z)$ is regular in $U$, $P(0) = 1$, and $\text{Re} P(z) > 0$ in $U$. From (3.8) and (3.9), we obtain

$$1 + \frac{zh''(z)}{h'(z)} = \frac{1}{2} + \frac{P(z)}{2} + \frac{zP'(z)}{P(z) + 3}. \quad \text{(3.10)}$$

Using Lemma 2.2 (with $\mu = 1/2$ and $\lambda = 3$), we obtain, after some calculations,

$$\text{Re} \left[ 1 + \frac{zh''(z)}{h'(z)} \right] \geq \begin{cases} 2 \frac{2}{(1+r)(2+r)}, & \text{if } 0 \leq r < -\frac{1 + \sqrt{5}}{2}, \\ 6 \left[ 2 - r^2 \right]^{1/2} - 2 \left( \frac{4 - 3r^2}{1 - r^2} \right), & \text{if } \frac{-1 + \sqrt{5}}{2} \leq r < 1, \end{cases}$$

(3.11)

where $|z| = r < 1$.

Now, we can easily check that for $0 \leq r < (-1 + \sqrt{5})/2$, $\text{Re}(1 + zh''(z)/h'(z)) > -1/2$ and for $(-1 + \sqrt{5})/2 \leq r < 1$, $\text{Re}(1 + zh''(z)/h'(z)) > -1/2$ whenever $23r^4 - 102r^2 + 63 > 0$ or whenever $r < r_1$, where $r_1 = ((51 - 24\sqrt{2})/23)^{1/2}$ is the smallest positive root of $23r^4 - 102r^2 + 63 = 0$. The desired result now follows from Lemma 2.1. □

In the following theorem, we take for $g$ some distinguished members of $K$.

**Theorem 3.3.** Let $f$ be regular in $U$ with $f(0) = 0$ and let $f(z) + zf'(z) < g(z) + zg'(z)$ in $U$. Then

(a) $f(z) < g(z)$ in $U$ if $g(z) = z/(1-z)$;
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(b) \( f(z) < g(z) \) at least in \(|z| < \rho_1 = \left(\frac{(28 - 8\sqrt{7})}{7}\right)^{1/2} = 0.98\) if \( g(z) = -\log(1-z) \);

(c) \( f(z) < g(z) \) in \( U \) if \( g(z) = z + \lambda z^2, |\lambda| \leq 1/5 \);

(d) \( f(z) < g(z) \) at least in \(|z| < \rho_2 = (9 - \sqrt{33})/4 = 0.8138\) if \( g(z) = e^z - 1 \).

We observe that the functions \( g \) defined in (a), (b), (c), and (d) belong to \( K, K(1/2), K(1/3), \) and \( K \), respectively.

**Proof.** We omit the proofs of parts (a), (c), and (d) and proceed to prove part (b). Let \( h(z) = g(z) + zg'(z) \), where \( g(z) = -\log(1-z) \). Then \( h(0) = 0 \) and \( h'(0) \neq 0 \). A simple computation shows that the condition

\[
\text{Re} \left( 1 + \frac{2h''(z)}{h'(z)} \right) > -\frac{1}{2}
\]

is equivalent to

\[
\text{Re} \left[ \frac{2}{(1-z)(2-z)} + \frac{1}{2} \right] > 0.
\]

If we let \( z = re^{i\theta}, 0 \leq r < 1 \) and \( 0 \leq \theta \leq 2\pi \), then condition (3.13) takes the form

\[
\varphi(x) = 16r^2 x^2 - 6r (4 + 2r^2) x + r^4 + r^2 + 12 > 0,
\]

where \( x = \cos \theta, 0 \leq \theta \leq 2\pi \). For \( r = 0 \), (3.14) is obviously satisfied. We, therefore, let \( r \neq 0 \). Now, it can be readily verified that at \( x = x_0 = (12 + 3r^2)/16r \), we have \( \varphi'(x) = 0 \) and \( \varphi''(x) > 0 \).

Thus, \( x = x_0 \) gives the minimum value of \( \varphi(x) \) provided \(-1 \leq x_0 \leq 1 \). This is true if \( r \geq \rho_0 = (8 - \sqrt{28})/3 = 0.9028\)…. Therefore, for \( r \in [\rho_0, 1] \),

\[
\min_{x \in [-1, 1]} \varphi(x) = \varphi(x_0) = \frac{7r^4 - 56r^2 + 48}{16}.
\]

Hence, in this case, (3.14) is satisfied if \( 7r^4 - 56r^2 + 48 > 0 \), i.e., if \( r < \rho_1 = ((28 - 8\sqrt{7})/7)^{1/2} = 0.98\)…. Also, for \( r \in [0, \rho_0] \), we can easily verify that \( \varphi(x) \) is a decreasing function of \( x \). Hence, in this case,

\[
\min_{x \in [-1, 1]} \varphi(x) = \varphi(1) = 4 - 6^3 + 17^2 - 24 + 12
\]

\[
= (1-r)(2-r)(r^2 - 3r + 6) > 0.
\]

Therefore, we conclude that for \( 0 \leq r < \rho_1 \),

\[
\text{Re} \left( 1 + \frac{2h''(z)}{h'(z)} \right) > -\frac{1}{2}.
\]

Conclusion (b) now follows in view of Lemma 2.1. \( \square \)

**References**


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Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

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