SOME SUFFICIENT CONDITIONS FOR STRONGLY STARLIKENESS

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Abstract. We consider strongly starlikeness of order $\alpha$, $0 < \alpha \leq 1$, which are analytic in the unit disc and satisfy the condition of the form

$$\left| f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \lambda, \quad 0 < \mu < 1, \ 0 < \lambda < 1.$$ 

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1. Introduction and preliminaries. Let $H$ denote the class of functions analytic in the unit disc $U = \{ z : |z| < 1 \}$ and let $A \subset H$ be the class of normalized analytic functions $f$ in $U$ such that $f(0) = f'(0) - 1 = 0$. For $n \geq 1$ we put

$$A_n = \{ f : f(z) = z + a_{n+1}z^{n+1} + \cdots \text{ is analytic in } U \} \quad (1.1)$$

and $A_1 = A$.

A function $f \in A$ is said to be strongly starlike of order $\alpha$, $0 < \alpha \leq 1$, if and only if

$$zf'(z) < \left( \frac{1+z}{1-z} \right)^{\alpha}, \quad (1.2)$$

where $<$ denotes the usual subordination. We denote this class by $S(\alpha)$. If $\alpha = 1$, then $S(1) \equiv S^*$ is the well-known class of starlike functions in $U$ (cf. [1]).

In this paper, we find a condition so that $f \in A_n$ satisfying

$$f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} < 1 + \lambda z, \quad 0 < \mu < 1, \ 0 < \lambda < 1, \quad (1.3)$$

is in $S(\alpha)$. Also, we consider an integral transformation.

We note that the author in [4] determined the values for $\lambda$ in (1.3) which implies starlikeness in $U$. Recently, Ponnusamy and Singh [5] found the condition which implies the strongly starlikeness of order $\alpha$, but for $\mu < 0$ in (1.3). By using the similar method as in [5] we consider strongly starlikeness in the case (1.3).

First, we cite the following lemma.

**Lemma 1.1.** Let $Q \in H$ satisfy the subordination condition

$$Q(z) < 1 + \lambda_1 z, \quad Q(0) = 1, \quad (1.4)$$

where $0 < \lambda_1 \leq 1$. For $0 < \alpha \leq 1$, let $p \in H$, $p(0) = 1$ and $p$ satisfy the condition

$$Q(z)p^{\alpha}(z) < 1 + \lambda z, \quad 0 < \lambda \leq 1. \quad (1.5)$$
Then for
\[ \sin^{-1} \lambda + \sin^{-1} \lambda_1 \leq \frac{\alpha \pi}{2} \] (1.6)
we have \( \text{Re}\{p(z)\} > 0 \) in \( U \).

This is the special case of the more general lemma given in [5].

2. Results and consequences. For our results we also need the following two lemmas.

**Lemma 2.1.** Let \( p \in H, \ p(z) = 1 + p_n z^n + \cdots, \ n \geq 1, \) satisfy the condition
\[ p(z) - \frac{1}{\mu} z p'(z) < 1 + \lambda z, \quad 0 < \mu < 1, \ 0 < \lambda \leq 1. \] (2.1)

Then
\[ p(z) \prec 1 + \lambda_1 z, \] (2.2)
where
\[ \lambda_1 = \frac{\lambda \mu}{n - \mu}. \] (2.3)

The proof of this lemma for \( n = 1 \) is given by [4]. For any \( n \in \mathbb{N} \) we also can apply Jack’s lemma [3].

**Lemma 2.2.** If \( 0 < \mu < 1, \ 0 < \lambda \leq 1 \) and \( Q \in H \) satisfying
\[ Q(z) < 1 + \frac{\lambda \mu}{n - \mu} z, \quad Q(0) = 1, \quad n \in \mathbb{N}, \] (2.4)
and if \( p \in H, \ p(0) = 1 \) and satisfies
\[ Q(z) p^n(z) < 1 + \lambda z, \] (2.5)
where
\[ 0 < \lambda \leq \frac{(n - \mu) \sin(\pi \alpha/2)}{[\mu + (n - \mu) e^{i \pi \alpha/2}]} \] (2.6)
then \( \text{Re}\{p(z)\} > 0 \) in \( U \).

**Proof.** If in Lemma 1.1 we put \( \lambda_1 = \lambda \mu / (n - \mu) \), then the condition (1.6) is equivalent to
\[ \sin^{-1} \lambda + \sin^{-1} \frac{\lambda \mu}{n - \mu} \leq \frac{\alpha \pi}{2}. \] (2.7)

This inequality is equivalent to
\[ \sin^{-1} \left( \sqrt{\frac{\lambda^2 \mu^2}{(n - \mu)^2} + \frac{\lambda \mu}{n - \mu} \sqrt{1 - \lambda^2}} \right) \leq \frac{\alpha \pi}{2}, \] (2.8)
or to the inequality
\[
\lambda \left[ \sqrt{(n-\mu)^2 - \lambda^2 \mu^2} + \mu \sqrt{1-\lambda^2} \right] \leq (n-\mu) \sin \left( \frac{\alpha \pi}{2} \right). \tag{2.9}
\]

From there, after some transformations, we get the following equivalent inequality
\[
\left\{ \left[ \mu^2 + (n-\mu)^2 \right]^2 - 4\mu^2 (n-\mu)^2 \cos^2 \left( \frac{\alpha \pi}{2} \right) \right\} \lambda^4
- 2(n-\mu)^2 \left[ \mu^2 + (n-\mu)^2 \right] \sin^2 \left( \frac{\alpha \pi}{2} \right) \lambda^2 + (1-\mu)^4 \sin^4 \left( \frac{\alpha \pi}{2} \right) \geq 0
\tag{2.10}
\]
which is equivalent to the condition (2.6).

By Lemma 1.1 we have that $\text{Re}\{p(z)\} > 0$ in $U$.

**Theorem 2.3.** Let $f \in A_n$, $0 < \mu < 1$ and $f$ satisfy the subordination
\[
f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} < 1 + \lambda z, \tag{2.11}
\]
where
\[
0 < \lambda \leq \frac{n-\mu}{\sqrt{\mu^2 + (n-\mu)^2}}. \tag{2.12}
\]
Then $f \in S^*$.

**Proof.** If we put $Q(z) = (z/f(z))^\mu = 1 + q_n z^n + \cdots$, then after some calculations, we get
\[
Q(z) - \frac{1}{\mu} z Q'(z) = f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} < 1 + \lambda z. \tag{2.13}
\]
From Lemma 2.1 we have
\[
Q(z) < 1 + \lambda_1 z, \quad \lambda_1 = \frac{\lambda \mu}{n-\mu}. \tag{2.14}
\]
The rest part of the proof is the same as in the case $n = 1$ (see [4, Theorem 1]) and we omit the details.

**Theorem 2.4.** Let $0 < \mu < 1$ and $0 < \alpha \leq 1$. If $f \in A_n$ satisfies
\[
\left| f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \frac{(n-\mu) \sin(\pi \alpha/2)}{\mu + (n-\mu)e^{i\pi \alpha/2}}, \quad z \in U, \tag{2.15}
\]
then $f \in S(\alpha)$.

**Proof.** If we put $\lambda = (n-\mu) \sin(\pi \alpha/2)/\mu + (n-\mu)e^{i\pi \alpha/2}$, then, since $0 < \alpha \leq 1$, we have $0 < \lambda \leq (n-\mu)/\sqrt{\mu^2 + (n-\mu)^2}$, and by Theorem 2.3, $f \in S^*$. Later, the function $Q(z) = (z/f(z))^\mu = 1 + q_n z^n + \cdots$ is analytic in $U$ and satisfies $Q(z) < 1 + \lambda_1 z$, $\lambda_1 = \lambda \mu/(n-\mu)$. Now, if we define
\[
p(z) = \left( \frac{zf'(z)}{f(z)} \right)^{1/\alpha}, \tag{2.16}
\]
then $p$ is analytic in $U$, $p(0) = 1$ and condition (2.15) is equivalent to

$$Q(z)p^\alpha(z) < 1 + \lambda z.$$  

(2.17)

Finally, from Lemma 2.2 we obtain

$$\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}} < \frac{1 + z}{1 - z} \iff \frac{zf'(z)}{f(z)} < \left(\frac{1 + z}{1 - z}\right)^\alpha,$$  

(2.18)

that is, $f \in S(\alpha)$.

We note that for $\alpha = 1$ we have the statement of Theorem 2.3.

For $n = 1$, $\mu = 1/2$, $\alpha = 2/3$ we get the following corollary.

**Corollary 2.5.** Let $f \in A$ and let

$$\left| f'(z) \left(\frac{z}{f(z)}\right)^{\frac{3}{2}} - 1 \right| < \frac{1}{2}, \quad z \in U.$$  

(2.19)

Then

$$\left| \arg \left(\frac{zf'(z)}{f(z)}\right) \right| < \frac{\pi}{3}, \quad z \in U,$$  

(2.20)

that is, $f \in S(2/3)$.

**Theorem 2.6.** Let $0 < \mu < 1$, $\text{Re}\{c\} > -\mu$, and $0 < \alpha \leq 1$. If $f \in A_n$ satisfies

$$\left| f'(z) \left(\frac{z}{f(z)}\right)^{1 + \mu} - 1 \right| < \left| \frac{n + c - \mu}{c - \mu} \right| \frac{(n - \mu) \sin(\pi \alpha/2)}{|\mu + (n - \mu) e^{i \pi \alpha/2}|}, \quad z \in U,$$  

(2.21)

then the function

$$F(z) = z \left[ \frac{c - \mu}{z - \mu} \int_0^z \left(\frac{t}{f(t)}\right)^\mu t^{c-\mu-1} dt \right]^{-1/\mu}$$  

(2.22)

belongs to $S(\alpha)$.

**Proof.** If we put that $\lambda$ is equal to the right-hand side of the inequality (2.21) and

$$Q(z) = F'(z) \left(\frac{z}{F(z)}\right)^{1 + \mu} (= 1 + q_nz^n + \cdots),$$  

(2.23)

then from (2.21) and (2.22) we obtain

$$Q(z) + \frac{1}{c - \mu} zQ'(z) = f'(z) \left(\frac{z}{f(z)}\right)^{1 + \mu} < 1 + \lambda z.$$  

(2.24)

Hence, by using the result of Hallenbeck and Ruscheweyh [2, Theorem 1] we have that

$$Q(z) < 1 + \lambda_1 z, \quad \lambda_1 = \frac{|(c - \mu)| \Lambda}{|n + c - \mu|} = \frac{(n - \mu) \sin(\pi \alpha/2)}{|\mu + (n - \mu) e^{i \pi \alpha/2}|},$$  

(2.25)

and the desired result easily follows from Theorem 2.4.
**Remark 2.7.** For $\alpha = 1$ and $n = 1$ we have the corresponding result given earlier in [4]. For $c = \mu + 1$, we have

**Corollary 2.8.** Let $0 < \mu < 1$ and $0 < \alpha \leq 1$. If $f \in A_n$ satisfies the condition

\[
\left| f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \frac{n(n-\mu) \sin(\pi \alpha/2)}{\mu + (n-\mu) e^{\pi \alpha/2}}, \quad z \in U,
\]

then the function

\[
F(z) = z \left[ \frac{1}{z} \int_0^z \left( \frac{t}{f(t)} \right)^\mu dt \right]^{-1/\mu}
\]

belongs to $S(\alpha)$.

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**References**


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<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>May 1, 2009</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>August 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>November 1, 2009</td>
</tr>
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