MULTIMODAL CYCLES WITH LINEAR MAP HAVING EXACTLY ONE FIXED POINT

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Abstract. We describe a class of cycles that cannot be forced by a cycle whose linear map has exactly one fixed point.

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1. Introduction. This note is concerned with the forcing relation on cycles. In particular, we consider cycles $\theta$ for which the $\theta$-linear map has exactly one fixed point. We prove a theorem which describes a large class of cycles that cannot be forced by $\theta$.

2. Definitions. Throughout this note, $f : I \rightarrow I$ denotes a continuous map of a compact interval. For $x \in I$, $f^0(x) = x$, and for $n \in \mathbb{N}$, $f^n(x) = f(f^{n-1}(x))$. An element $x \in I$ is a periodic point for $f$ if there exists $k \in \mathbb{N}$ satisfying $f^k(x) = x$. The least such $k$ is called the period of $x$. A point of period 1 is called a fixed point. The orbit of $x \in I$ is the set $\{f^n(x)\}_{n=0}^\infty$ and is denoted $\mathcal{O}(x)$. If $x$ is periodic with period $k$, then $\mathcal{O}(x)$ is a finite set consisting of $k$ distinct elements.

A cycle of order $n$ is a bijection $\theta : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ satisfying $\theta^k(1) \neq 1$ for $1 \leq k < n$. Let $x$ be a periodic point for $f$ with least period $n$ and $\mathcal{O}(x) = \{x_1 < x_2 < \cdots < x_n\}$. We say that $x$ has orbit type $\theta$ if $\theta$ is a cycle of order $n$ and $f(x_i) = x_{\theta(i)}$ for $1 \leq i \leq n$. In this case, we also say that the periodic orbit $\mathcal{O}(x)$ has orbit type $\theta$. We say that $f$ has a periodic orbit of orbit type $\theta$ if there exists a periodic point $x \in I$ which has orbit type $\theta$. A cycle $\theta$ forces a cycle $\eta$ if whenever $f$ has a periodic orbit of type $\theta$, $f$ has a periodic point of type $\eta$.

For a cycle $\theta$ of order $n$, the $\theta$-linear map $L_\theta : [1, n] \rightarrow [1, n]$ is defined by

$$L_\theta(k) = \theta(k), \quad \text{for } 1 \leq k \leq n,$$

$$L_\theta \text{ is linear on } [i, i+1], \quad \text{for } 1 \leq i \leq n-1. \quad (2.1)$$

The graph of $L_\theta$ consists of at most $n-1$ linear segments, each having a slope $m$ satisfying $|m| \geq 1$. A cycle $\eta$ is forced by $\theta$ if and only if $L_\theta$ has a periodic orbit of type $\eta$ [1].

Baldwin [2] defined the forcing relation and proved that the forcing relation induces a partial order on the set of cycles. He provided an exhaustive but inefficient algorithm for determining whether one cycle forces another. Jungreis [6] provided a combinatorial method to determine if one cycle forces another in certain cases. In [3] a geometric version of Jungreis’s algorithm is given and in [4] this algorithm is generalized to any
two cycles. In [8], another geometric algorithm is given to determine the forcing relation. This algorithm is similar to Baldwin’s original algorithm but more efficient. A cycle is called unimodal if $L_\theta$ has exactly one turning point (a maximum, say). In [5] the forcing relation on the set of unimodal cycles is studied. In particular, it is shown that the forcing relation induces a total order on the set of unimodal cycles. In [7, 9] the structure of this totally ordered set is investigated.

3. Preliminaries. In this section, we define the $RL$-pattern for any cycle, and we define the step number for a cycle $\theta$ for which $L_\theta$ has exactly one fixed point.

**Definition 3.1.** Let $\eta$ be any cycle of order $k$. The $RL$-pattern for $\eta$ is the sequence

$$G = G_1G_2 \cdots G_k \in \{R, L\}^k$$  \hspace{1cm} (3.1)

defined by

$$G_i = \begin{cases} R & \text{if } \eta^i(1) > \eta^{i-1}(1), \\ L & \text{if } \eta^i(1) < \eta^{i-1}(1). \end{cases}$$  \hspace{1cm} (3.2)

Let $R(\eta)$ denote the length of the longest string of consecutive $R$’s in the $RL$-pattern for $\eta$.

Obviously, every $RL$-pattern begins with an $R$ and ends with an $L$.

Let $\theta$ be a cycle of order $n$ such that $L_\theta$ has exactly one fixed point. Let $p_1 \in (1, n)$ denote the unique fixed point and let $E_1 = \{x < p_1 \mid f(x) = p_1\}$. If $E_1 \neq \emptyset$, we let $p_2 = \max \{E_1\}$. For $i > 1$, if the points $p_1, p_2, \ldots, p_i$ and nonempty sets $E_1, \ldots, E_{i-1}$ have been defined, we set

$$E_i = \{x < p_i \mid f(x) = p_i\}.$$  \hspace{1cm} (3.3)

If $E_i \neq \emptyset$, we let $p_{i+1} = \max \{E_i\}$. We see that for some $i \geq 1$, $E_i = \emptyset$, for otherwise, there would exist a strictly decreasing sequence $\{p_n\}_{n=1}^\infty$ in $[1, n]$, converging to a point $p < p_1$ but satisfying, for each $n$,

$$L_\theta(p_n) = p_{n-1},$$  \hspace{1cm} (3.4)

so that by continuity,

$$\lim_{n \to \infty} L(p_n) = L(p)$$  \hspace{1cm} (3.5)

and at the same time

$$\lim_{n \to \infty} L(p_n) = \lim_{n \to \infty} p_{n-1} = p.$$  \hspace{1cm} (3.6)

Thus $L(p) = p$, which would contradict the assumption that $L_\theta$ has exactly one fixed point. Therefore we can make the following definition.

**Definition 3.2.** Let $\theta$ be a cycle of order $n$ such that $L_\theta$ has exactly one fixed point. The step number of $\theta$, denoted $S(\theta)$, is the (smallest) value of $i$ for which $E_i = \emptyset$.

**Example 3.3.** The cycle $\eta_1 = (1 \ 2 \ 3 \ 4)$ has $RL$-pattern $RRRL$. The cycle $\eta_2 = (1 \ 4 \ 7 \ 2 \ 6 \ 8 \ 5)$ has $RL$-pattern $RRLRLRLL$; $R(\eta_1) = 3$ and $R(\eta_2) = 2$. 

4. Results. For any cycle $\theta$ such that $L_\theta$ has exactly one fixed point, the following theorem describes a large class of cycles that cannot be forced by $\theta$.

**Theorem 4.1.** Let $\theta$ be a cycle of order $n \geq 2$ such that $L_\theta$ has exactly one fixed point. Let $S(\theta)$ denote the step number of $\theta$. Let $\eta$ be any cycle. If $R(\eta) > S(\theta)$, then $\theta$ does not force $\eta$.

**Proof.** We have

\[ 1 < p_{S(\theta)} < p_{S(\theta)-1} < \cdots < p_2 < p_1 < n. \]  

(4.1)

We write

\[ [1,n] = \bigcup_{i=1}^{S(\theta)+1} I_i, \]  

(4.2)

where

\[ I_1 = [p_1,n], \]
\[ I_i = [p_i,p_{i-1}] \quad \text{for } 2 \leq i \leq S(\theta), \]
\[ I_{S(\theta)+1} = [1,p_{S(\theta)}]. \]  

(4.3)

For any $x \in \text{int}(I_1)$, $L_\theta(x) < x$. So $x$ cannot be the leftmost point in any periodic orbit. For $2 \leq i \leq S(\theta) + 1$, we argue inductively. If $x \in \text{int}(I_i)$, then $L_\theta(x) > x$ and $L_\theta(x) \in \bigcup_{j=1}^{i-1} I_j$, so if $x$ is the leftmost point of a periodic orbit of type $\gamma$, the RL-pattern of $\gamma$ consist of at most $i-1$ consecutive $R$’s followed by an $L$. That is, $R(\gamma) \leq i-1$. This shows that any cycle $\eta$ forced by $\theta$ must have $R(\eta) \leq S(\theta)$.

**Example 4.2.** Let $\theta = (1 \ 2 \ 6 \ 3 \ 4 \ 5)$. $L_\theta$ has exactly one fixed point and $S(\theta) = 3$. From Theorem 4.1, we know that for all $n \geq 5$, $\theta$ does not force $(1 \ 2 \ 3 \cdots n)$. Using the technique developed in [8] it is seen that $\theta$ does force $(1 \ 2 \ 3 \ 4)$ and that there are exactly two distinct orbits of type $(1 \ 2 \ 3 \ 4)$. Also, $\theta$ forces $(1 \ 2 \ 3)$ and there are six distinct orbits of type $(1 \ 2 \ 3)$.

**Example 4.3.** Let $\theta = (1 \ 3 \ 5 \ 2 \ 8 \ 4 \ 7 \ 6)$. $L_\theta$ has one fixed point and $S(\theta) = 2$. From Theorem 4.1, we see that for all $n \geq 4$, $\theta$ does not force $(1 \ 2 \ 3 \cdots n)$. Using [8], one can find exactly two distinct orbits of type $(1 \ 2 \ 4 \ 3)$, exactly fourteen distinct orbits of type $(1 \ 3 \ 2 \ 4)$, exactly eleven distinct orbits of type $(1 \ 4 \ 2 \ 3)$ and one can show that there are now orbits of type $(1 \ 3 \ 4 \ 2)$ and no orbits of type $(1 \ 4 \ 3 \ 2)$. These are the only orbit types of period 4 forced by $\theta$.

**References**


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