STRUCTURE OF WEAKLY PERIODIC RINGS WITH POTENT EXTENDED COMMUTATORS

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Dedicated to the memory of Professor Hisao Tominaga

ABSTRACT. A well-known theorem of Jacobson (1964, page 217) asserts that a ring \( R \) with the property that, for each \( x \) in \( R \), there exists an integer \( n(x) > 1 \) such that \( x^{n(x)} = x \) is necessarily commutative. This theorem is generalized to the case of a weakly periodic ring \( R \) with a "sufficient" number of potent extended commutators. A ring \( R \) is called weakly periodic if every \( x \) in \( R \) can be written in the form \( x = a + b \), where \( a \) is nilpotent and \( b \) is "potent" in the sense that \( b^{n(b)} = b \) for some integer \( n(b) > 1 \). It is shown that a weakly periodic ring \( R \) in which certain extended commutators are potent must have a nil commutator ideal and, moreover, the set \( N \) of nilpotents forms an ideal which, in fact, coincides with the Jacobson radical of \( R \).

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1. Introduction. A ring \( R \) is called periodic if for each \( x \) in \( R \) there exist distinct positive integers \( m \) and \( n \) such that \( x^m = x^n \). An element \( x \) is called potent if for some integer \( n = n(x) > 1 \), \( x^n = x \). \( R \) is called weakly periodic if every \( x \) in \( R \) can be written (not necessarily uniquely) as a sum of a nilpotent element and a potent element. It is well known that a periodic ring is necessarily weakly periodic (see [2]). Whether a weakly periodic ring is necessarily periodic is apparently not known. Moreover, by a theorem of Chacron (see [3]), \( R \) is periodic if and only if for each \( x \) in \( R \), there exists a positive integer \( k = k(x) \) and a polynomial \( f(\lambda) = f_x(\lambda) \) with integer coefficients such that \( x^k = x^{k+1} f(x) \). For \( x, y \) in \( R \), \([x,y]_1 = [x,y] = xy - yx \) denotes the usual commutator, and for every positive integer \( k > 1 \), we define the extended commutator \([x,y]_k\) inductively by \([x,y]_k = [[x,y]_{k-1},y] \).

2. Main results. We begin with some basic facts about weakly periodic rings.

**Lemma 2.1.** The homomorphic image of any weakly periodic ring is weakly periodic.

This follows readily from the definition of a weakly periodic ring.

**Lemma 2.2.** A weakly periodic division ring is necessarily commutative.

This follows from the "\( x^{n(x)} = x \)" theorem of Jacobson (see [5]).

**Lemma 2.3.** Let \( R \) be a weakly periodic ring, \( N \) the set of nilpotents, and \( J \) the Jacobson radical of \( R \). Then \( J \subseteq N \).
Proof. Suppose $j \in J$. Then,

$$j = a + b, \quad a \in N, \quad b^n = b \quad \text{for some } n > 1.$$  \hfill (2.1)

Suppose $a^q = 0$. Then,

$$(j - b)^{(n-1)q+1} = a^{(n-1)q+1} = 0,$$  \hfill (2.2)

which implies (since $j \in J$) $b^{(n-1)q+1} \in J$. But $b^{(n-1)q+1} = b$, since $b^m = b$, and hence $b \in J$. Since $b^{n-1}$ is an idempotent element in $J$, $b^{n-1} = 0$. Therefore, $b = b^m = 0$, and hence by (2.1), $j = a \in N$. Thus, $J \subseteq N$.

Theorem 2.4. Let $R$ be a weakly periodic ring and suppose that $N$ is the set of nilpotents of $R$. Let $n > 1$ be a fixed integer. Suppose that for all $x_1, \ldots, x_n$ in $R \backslash N$, $\sigma$ is a permutation in $S_n$ such that $\sigma(n) \neq n$. Suppose, further, that for all $x_1, \ldots, x_n$ in $R \backslash N$, there exists a positive integer $k$ such that the following extended commutator is potent, namely

$$[x_1 \cdots x_n, x_{\sigma(1)} \cdots x_{\sigma(n)}]_k \text{ is potent, } \forall x_i \in R \backslash N.$$  \hfill (2.3)

Then,

(i) The commutator ideal of $R$ is nil, $(C(R) \subseteq N)$.

(ii) $N$ is an ideal of $R$.

(iii) $N = J$, the Jacobson radical of $R$.

(iv) $R$ is periodic.

Proof. The semisimple ring $R/J$ is isomorphic to a subdirect sum of primitive rings $R_i$ ($i \in \Gamma$). By Lemma 2.1, each $R_i$ is a weakly periodic ring. Now, by Jacobson’s density theorem, we must have

(a) $R_i$ is a division ring, or

(b) for some positive integer $m > 1$, there exists a subring $T_i$ of $R_i$ which maps homomorphically onto $D_m$, the complete matrix ring of $m \times m$ matrices over some division ring $D$.

In case (a), $R_i$ is commutative, by Lemma 2.2. In case (b), $D_m$ must satisfy (2.3), since (2.3) is inherited by all subrings and all homomorphic images of $R$, where $m > 1$. The net result is:

(*) The ring $D_m$ of all $m \times m$ matrices over the division ring $D$ satisfies (2.3), where $m > 1$.

That statement (*) is false can be seen by taking

$$x_1 = x_2 = \cdots = x_{n-1} = E_{11}, \quad x_n = E_{11} + E_{12}.$$  \hfill (2.4)

In verifying this, note that

$$x_1 \cdots x_n = E_{11} (E_{11} + E_{12}) = E_{11} + E_{12};$$

$$x_{\sigma(1)} \cdots x_{\sigma(n)} = (E_{11} + E_{12})E_{11} = E_{11};$$

$$[x_1 \cdots x_n, x_{\sigma(1)} \cdots x_{\sigma(n)}] = -E_{12};$$

$$[x_1 \cdots x_n, x_{\sigma(1)} \cdots x_{\sigma(n)}]_k = (-1)^k E_{12},$$

which is not potent. This contradiction shows that case (b) never occurs, and hence
Each $R_i$ is a commutative division ring (case (a)). Thus, $R/J$ is indeed commutative, and hence

$$C(R) \subseteq J, \quad (C(R) \text{ denotes the commutator ideal of } R). \quad (2.6)$$

Combining this with Lemma 2.3, we see that $C(R) \subseteq N$, which proves (i). Part (ii) follows at once from part (i) by considering the commutative ring $R/C(R)$. To prove (iii), observe that $N \subseteq J$ (since $N$ is an ideal) while $J \subseteq N$, by Lemma 2.3. Thus, $N = J$.

To prove part (iv), let $x \in R$. Then, by definition of weakly periodic ring, there exist elements $a$ and $b$ such that

$$x = a + b, \quad a \in N; \quad b^\gamma = b, \quad \text{for some } \gamma > 1. \quad (2.7)$$

Hence, since $N$ is an ideal (part (ii)) and $a \in N$,

$$x - a = b = b^\gamma = (x - a)^\gamma = x^\gamma + a_0 \quad (a_0 \in N). \quad (2.8)$$

Therefore, $x - x^\gamma = a + a_0 \in N$, and thus $(x - x^\gamma)^\alpha = 0$ for some positive integer $\alpha$. Hence, $x^\alpha = x^{\alpha+1} f(x)$, $f(\lambda) \in \mathbb{Z} \{\lambda\}$, and thus by Chacron’s theorem (see Section 1) $R$ is periodic. This proves the theorem. □

In preparation for the proof of the next theorem, we first prove the following lemmas.

**Lemma 2.5.** Let $R$ be an arbitrary ring (not necessarily weakly periodic), and suppose $N$ is the set of nilpotents of $R$. Let $n > 1$ be a fixed integer. Suppose that for all $x_1, \ldots, x_n$ in $R \setminus N$, $\sigma$ is a permutation in $S_n$ such that $\sigma(1) \neq 1$ and $\sigma(n) \neq n$. Suppose, further, that for all $x_1, \ldots, x_n$ in $R \setminus N$, there exists a positive integer $k$ such that

$$[x_1 \cdots x_n, x_{\sigma(1)} \cdots x_{\sigma(n)}]_k \text{ is potent,} \quad \forall x_i \in R \setminus N. \quad (2.9)$$

Then, the set $E$ of idempotents of $R$ is central.

**Proof.** Suppose $e \in E$, $x \in R$, $a = ex - exe$, $f = e + a$. We claim that $ef = fe$. Suppose $ef \neq fe$, then $e \neq 0$, $f \neq 0$, and (since $e^2 = e$, $f^2 = f$) hence $e \notin N$, $f \notin N$. Therefore, by (2.9) with $x_1 = \cdots = x_{n-1} = e$, $x_n = f$, we have $[ef, fe]_k$ is potent, and hence $[f, e]_k = (-1)^k a$ is potent. Since $a^2 = 0$, it follows that $a = 0$. Hence, $f = e + a = e$, which contradicts the hypothesis $ef \neq fe$. This contradiction shows that $ef = fe$, and hence $e(e + a) = (e + a)e$. Thus, $a = ea = ae = 0$. Hence, $exe = exe$. A similar argument, using $a’ = xe - exe$, $f’ = e + a’$ shows that $xe = exe$, and hence $exe = exe$. This proves the lemma. □

**Lemma 2.6.** Suppose that $R$ is a weakly periodic ring which satisfies the hypotheses of Theorem 2.4. Suppose $\delta : R \to R^*$ is a homomorphism of $R$ onto $R^*$, and let $N$ be the set of nilpotents of $R$. Then, the set $N^*$ of nilpotents of $R^*$ coincides with the set $\delta(N)$.

**Proof.** By Theorem 2.4(iv), $R$ is periodic. The lemma now follows from [1]. □

**Lemma 2.7.** Let $R$ be a subdirectly irreducible ring. Then, the only central idempotents of $R$ are 0 and 1 (if 1 $\in R$).

This lemma is well known, and we omit the proof.
Lemma 2.8. Let \( R \) be a ring, and let \( x, y \in R \). Suppose that \([x, y]\) commutes with \( x \). Then, for all positive integers \( k \), we have

\[
[x^k, y] = kx^{k-1}[x, y].
\] (2.10)

(Equivalently, \([y, x^k] = kx^{k-1}[y, x]\).)

This follows at once, by induction.

Lemma 2.9. Let \( R \) be a periodic ring with the set \( N \) of nilpotents commutative. If for each \( a \in N \) and \( x \in R \) there exists a positive integer \( k \) such that \([a, x]_k = 0\), then \( R \) is commutative.

This lemma was proved by Bell [4].

We are now in a position to prove our next theorem.

Theorem 2.10. Let \( R \) be a weakly periodic ring, and let \( N \) denote the set of nilpotents of \( R \). Let \( n > 1 \) be a fixed integer. Suppose that for all \( x_1, \ldots, x_n \) in \( R \setminus N \), \( \sigma \) is a permutation in \( S_n \) such that \( \sigma(1) = n \) and \( \sigma(n) = 1 \). Suppose that, for all \( x_1, \ldots, x_n \) in \( R \setminus N \), there exists a positive integer \( k \) such that

\[
[x_1 \cdots x_n, x_{\sigma(1)} \cdots x_{\sigma(n)}]_k \text{ is potent}, \quad \forall x_i \in R \setminus N. \tag{2.11}
\]

Suppose, further, that

\[
[a, b] \text{ is potent} \quad \forall a, b \in N. \tag{2.12}
\]

Then, \( R \) is commutative.

Proof. In view of Lemma 2.6, all the hypotheses are inherited by homomorphic images of \( R \); and since every ring is isomorphic to a subdirect sum of subdirectly irreducible rings, we may assume that \( R \) is subdirectly irreducible. Since \( N \) is an ideal, by Theorem 2.4(ii), we see that for all \( a, b \) in \( N \), \([a, b] \) is both potent (see (2.12)) and nilpotent, and hence \([a, b] = 0\), which implies that \( N \) is commutative.

Now, \( R \) is periodic, by Theorem 2.4(iv), and hence some power of each element of \( R \) is idempotent. Therefore, by Lemmas 2.5 and 2.7, either \( R \) is nil or \( R \) has an identity \( 1 \).

In the first case, \( R = N \) is commutative and there is nothing further to prove. So we assume that \( 1 \in R \).

Let \( a \in N \) and \( x \in R \setminus N \). Since \( 1 + a \notin N \), there exists a positive integer \( k \) such that

\[
[(1 + a) \cdot 1 \cdot 1 \cdots 1 \cdot x, x \cdot 1 \cdot 1 \cdots 1 \cdot (1 + a)]_k \text{ is potent},
\]

and thus \([x + ax, x + xa]_k \text{ is potent.} \tag{2.13}

Next, we show, by induction, that

\[
[x + ax, x + xa]_m = [a, x]_{m+1} \quad \text{for all positive integers } m. \tag{2.14}
\]

To begin with, observe that

\[
[x + ax, x + xa]_1 = [x, xa] + [ax, x] + [ax, xa]. \tag{2.15}
\]
Since $N$ is a commutative ideal of $R$ and $a \in N$, therefore $[ax,xa] = 0$, and hence (2.15) is equivalent to

$$[x + ax, x + xa]_1 = -[xa, x] + [ax, x] = [a, x]_2.$$ (2.16)

Hence (2.14) is true for $m = 1$. Now, suppose (2.14) is true for $m = q$. That is, suppose that

$$[x + ax, x + xa]_q = [a, x]_q + 1.$$ (2.17)

This induction hypothesis implies that

$$[x + ax, x + xa]_{q+1} = [[a, x]_{q+1}, x + xa]$$
$$= [[a, x]_{q+1}, x] + [[a, x]_{q+1}, xa]$$
$$= [[a, x]_{q+1}, x] = [a, x]_{q+2},$$

since $[[a, x]_{q+1}, xa] = 0$ (recall that $a \in N$ and $N$ is a commutative ideal of $R$). Thus, (2.14) is true for $m = q + 1$, completing this induction proof of (2.14). Now, combining (2.13) and (2.14), we see that

$$[a, x]_{k+1} \text{ is potent } (a \in N, x \in R).$$ (2.18)

Since $[a, x]_{k+1}$ is also in the ideal $N$, therefore this extended commutator is both nilpotent and potent, and hence

$$[a, x]_{k+1} = 0.$$ (2.19)

Keeping in mind that $R$ is periodic and $N$ is commutative, and combining (2.19) with Lemma 2.9, it follows that $R$ is commutative, and the theorem is proved.

**Corollary 2.11.** Suppose $R$ is a periodic ring with commuting nilpotents and with the property that, for all $x, y$ in $R$, there exists a positive integer $k$ such that $[xy, yx]_k = 0$. Then, $R$ is commutative.

**Proof.** Since $R$ is periodic, $R$ is also weakly periodic (see Section 1). Therefore, all the hypotheses of Theorem 2.10 are satisfied (take $n = 2$ in (2.11)), and hence $R$ is commutative.

The following is another corollary which yields a result proved by Putcha and Yaqub [6].

**Corollary 2.12.** A periodic ring with commuting nilpotents and central commutators is commutative.

**Proof.** Let $x, y$ be any elements of $R$. Since $[x, y]$ is in the center of $R$, therefore $[[x, y], y] = 0$; that is, $[x, y]_2 = 0$. Thus,

$$[x, y]_2 = 0 \ \forall x, y \in R.$$ (2.20)

In particular, $[xy, yx]_2 = 0$, and the corollary now follows by taking $k = 2$ in Corollary 2.11.
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REFERENCES


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<thead>
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<th>May 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>August 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>November 1, 2009</td>
</tr>
</tbody>
</table>

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