MIXED PROBLEM WITH NONLOCAL BOUNDARY CONDITIONS FOR A THIRD-ORDER PARTIAL DIFFERENTIAL EQUATION OF MIXED TYPE

M. DENCHE and A. L. MARHOUNE

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Abstract. We study a mixed problem with integral boundary conditions for a third-order partial differential equation of mixed type. We prove the existence and uniqueness of the solution. The proof is based on two-sided a priori estimates and on the density of the range of the operator generated by the considered problem.

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1. Introduction. In the rectangle $\Omega = (0, \ell) \times (0, T)$, we consider the equation

$$Lu = \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial^2 u}{\partial x \partial t} \right) = f(x, t),$$

(1.1)

where $a(x, t)$ is bounded with $0 < a_0 < a(x, t) \leq a_1$ and has bounded partial derivatives such that $0 < a_2 \leq \partial a(x, t)/\partial t \leq a_3$ and $0 < a_4 \leq \partial^2 a(x, t)/\partial x \partial t \leq a_5$ for $(x, t) \in \Omega$.

To (1.1) we add the initial conditions

$$l_1 u = u(x, 0) = \varphi(x), \quad l_2 u = \frac{\partial u}{\partial t}(x, 0) = \psi(x), \quad x \in (0, \ell),$$

(1.2)

the Dirichlet condition

$$u(0, t) = 0, \quad t \in (0, T),$$

(1.3)

and the integral condition

$$\int_0^\ell u(\xi, t) d\xi = 0, \quad t \in (0, T),$$

(1.4)

where $\varphi$ and $\psi$ are known functions which satisfy the compatibility conditions given by (1.3) and (1.4), that is,

$$\varphi(0) = 0, \quad \int_0^\ell \varphi(x) dx = 0, \quad \psi(0) = 0, \quad \int_0^\ell \psi(x) dx = 0.$$  

(1.5)

Boundary-value problems for parabolic equations with integral boundary conditions are investigated by Batten [1], Bouziani and Benouar [2], Cannon [3, 4], Perez Esteva and van der Hoeck [5], Ionkin [8], Kamynin [9], Kartynnik [10], Shil [11], Yurchuk [13], and many references therein. We remark that integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics; see for example, [6, 7, 11, 12].
The present paper is devoted to the study of a mixed problem with boundary integral conditions for a third-order partial differential equation of mixed type.

We associate to problem (1.1), (1.2), (1.3), and (1.4) the operator

\[ L = \left( \frac{\partial^2 u}{\partial t^2}, l_1, l_2 \right) \]

deﬁned from \( E \) into \( F \), where \( E \) is the Banach space of functions \( u \in L^2(\Omega) \), satisfying (1.3) and (1.4), with the ﬁnite norm

\[
\| u \|_E^2 = \int_{\Omega} (\ell - x)^2 \left[ \left| \frac{\partial^2 u}{\partial t^2} \right|^2 + \left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^2 \right] dx \, dt
+ \sup_{0 \leq t \leq T} \int_0^\ell (\ell - x)^2 \left[ \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial u}{\partial x} \right|^2 \right] dx
+ \sup_{0 \leq t \leq T} \int_0^\ell \int_0^\ell \left[ \left| \frac{\partial u}{\partial t} \right|^2 + |u|^2 \right] dx dt,
\]

and \( F \) is the Hilbert space of vector-valued functions \( \mathcal{F} = (f, \varphi, \psi) \) obtained by completion of the space \( L^2(\Omega) \times W^2_2(0, \ell) \times W^2_2(0, \ell) \) with respect to the norm

\[
\| \mathcal{F} \|_F^2 = \| (f, \varphi, \psi) \|_F^2
= \int_{\Omega} (\ell - x)^2 |f|^2 \, dx \, dt + \int_0^\ell (\ell - x)^2 \left[ \left| \frac{d\varphi}{dx} \right|^2 + \left| \frac{d\psi}{dx} \right|^2 \right] dx + \int_0^\ell \int_0^\ell \left[ |\varphi|^2 + |\psi|^2 \right] dx dt.
\]

Using the energy inequalities method proposed in [13], we establish two-sided a priori estimates. Then, we prove that the operator \( L \) is a linear homeomorphism between the spaces \( E \) and \( F \).

2. Two-sided a priori estimates

**Theorem 2.1.** For any function \( u \in E \), there is the a priori estimate

\[
\| Lu \|_F \leq c \| u \|_E,
\]

where the constant \( c \) is independent of \( u \).

**Proof.** Using (1.1) and the initial conditions (1.2), we obtain

\[
\int_{\Omega} (\ell - x)^2 |\mathcal{F} u|^2 \, dx \, dt \leq 3 \int_{\Omega} (\ell - x)^2 \left[ \left| \frac{\partial^2 u}{\partial t^2} \right|^2 + a_1^2 \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 + a_1^2 \left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^2 \right] dx \, dt,
\]

\[
\int_0^\ell (\ell - x)^2 \left[ \left| \frac{d\varphi}{dx} \right|^2 + \left| \frac{d\psi}{dx} \right|^2 \right] dx \leq \sup_{0 \leq t \leq T} \int_0^\ell (\ell - x)^2 \left[ \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial u}{\partial x} \right|^2 \right] dx.
\]

Combining the inequalities (2.2), we obtain (2.1) for \( u \in E \).

**Theorem 2.2.** For any function \( u \in E \), there is the a priori estimate

\[
\| u \|_E \leq \alpha \| Lu \|_F,
\]

with the constant

\[
\alpha = \max \left( \frac{167/10, a_1}{\min(\exp(-cT)/20, \exp(-cT) a_0^2/15)} \right),
\]

\[
(2.4)
\]
and $c$ is such that

$$c \geq 1, \quad ca_0 - 1 \geq a_3 + 2\alpha^2 s.$$  \hfill (2.5)

Before proving this theorem, we first give the following two lemmas.

**Lemma 2.3.** For $u \in E$ satisfying the first condition in (1.2),

$$\frac{1}{2} \int_0^\ell \int_0^\ell (\ell - x)^2 \exp(-ct) \left| \frac{\partial^2 u}{\partial x^2} \right|^2 dx dt + \frac{c-1}{2} \int_0^\ell \int_0^\ell (\ell - x)^2 \exp(-ct) \left| \frac{\partial u}{\partial x} \right|^2 dx dt \geq \frac{1}{2} \int_0^\ell (\ell - x)^2 \exp(-ct) \left| \frac{\partial u}{\partial x}(\ell,\tau) \right|^2 dx - \frac{1}{2} \int_0^\ell (\ell - x)^2 \left| \frac{\partial u}{\partial x} \right|^2 dx.$$  \hfill (2.6)

**Proof.** Starting from

$$\int_0^\tau \int_0^\ell (\ell - x)^2 \exp(-ct) \frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial t} dx dt,$$  \hfill (2.7)

then integrating by parts and using elementary inequalities, we obtain (2.6). \hfill \Box

**Lemma 2.4.** For $u \in E$ satisfying the initial conditions (1.2),

$$\int_0^\ell \exp(-c\tau) |u(x,\tau)|^2 dx \leq \int_0^\tau \int_0^\ell \exp(-ct) \left| \frac{\partial u}{\partial t} \right|^2 dx dt + \int_0^\ell |\varphi|^2 dx,$$  \hfill (2.8)

with $c \geq 1$.

**Proof.** Integrating by parts the expression

$$\int_0^\tau \int_0^\ell \exp(-ct) u \frac{\partial \varphi}{\partial t} dx dt$$  \hfill (2.9)

and using elementary inequalities yield (2.8). \hfill \Box

**Remark 2.5.** We note that Lemmas 2.3 and 2.4 hold for weaker conditions on $u$.

**Proof of Theorem 2.2.** First, define

$$D(L) = \left\{ u \in E \mid \frac{\partial^3 u}{\partial x^3 \partial t} \in L_2(\Omega) \right\}, \quad Mu = (\ell - x)^2 \frac{\partial^2 u}{\partial t^2} + 2(\ell - x)J \frac{\partial^2 u}{\partial t^2},$$  \hfill (2.10)

where

$$Ju = \int_0^x u(\xi, t) d\xi.$$  \hfill (2.11)

We consider for $u \in D(L)$ the quadratic formula

$$\text{Re} \int_0^\tau \int_0^\ell \exp(-ct) \langle \mathcal{F} u, \overline{M u} \rangle dx dt,$$  \hfill (2.12)

with the constant $c$ satisfying (2.5), obtained by multiplying (1.1) by $\exp(-ct)Mu$, by
integrating over $\Omega^\tau$, where $\Omega^\tau = (0, \ell) \times (0, \tau)$, with $0 \leq \tau \leq T$, and by taking the real part. Integrating by parts (2.12) with the use of boundary conditions (1.3) and (1.4), we obtain

\[
\text{Re} \int_0^T \int_0^\ell \exp(-ct) \mathcal{L} \mu M \alpha dx \, dt = \left( \int_0^T \int_0^\ell (\ell - x)^2 \exp(-ct) \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx \, dt \right. + \frac{1}{2} \int_0^T \int_0^\ell \exp(-ct) J \frac{\partial^2 u}{\partial t^2} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx \, dt
\]

\[
+ \text{Re} \int_0^T \int_0^\ell (\ell - x)^2 \exp(-ct) a \frac{\partial^2 u}{\partial x \partial t} \left( \frac{\partial^2 u}{\partial x \partial t} \right) dx \, dt
\]

\[
+ 2 \text{Re} \int_0^T \int_0^\ell \exp(-ct) \frac{\partial u}{\partial t} a \frac{\partial^2 u}{\partial t^2} dx \, dt
\]

\[
+ 2 \text{Re} \int_0^T \int_0^\ell \exp(-ct) \frac{\partial a}{\partial x} \frac{\partial u}{\partial t} J \frac{\partial^2 u}{\partial t^2} dx \, dt.
\] (2.13)

On the other hand, by using the elementary inequalities we get

\[
\text{Re} \int_0^T \int_0^\ell \exp(-ct) \mathcal{L} \mu M \alpha dx \, dt \geq \left( \int_0^T \int_0^\ell (\ell - x)^2 \exp(-ct) \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx \, dt \right. \]

\[
+ \text{Re} \int_0^T \int_0^\ell (\ell - x)^2 \exp(-ct) a \frac{\partial^2 u}{\partial x \partial t} \left( \frac{\partial^2 u}{\partial x \partial t} \right) dx \, dt
\]

\[
+ 2 \text{Re} \int_0^T \int_0^\ell \exp(-ct) \frac{\partial u}{\partial t} a \frac{\partial^2 u}{\partial t^2} dx \, dt
\]

\[
- 2 \text{Re} \int_0^T \int_0^\ell \exp(-ct) \frac{\partial a}{\partial x} \frac{\partial u}{\partial t} J \frac{\partial^2 u}{\partial t^2} dx \, dt.
\] (2.14)

Again, integrating by parts the second and third terms of the right-hand side of the inequality (2.14) and taking into account the initial conditions (1.2) give

\[
\text{Re} \int_0^T \int_0^\ell \exp(-ct) \mathcal{L} \mu M \alpha dx \, dt + \int_0^\ell a(x, 0) |\psi|^2 dx + \frac{1}{2} \int_0^\ell a(x, 0) (\ell - x)^2 \left| \frac{d\psi}{dx} \right|^2 dx \, dt
\]

\[
\geq \left( \int_0^T \int_0^\ell \exp(-ct) (\ell - x)^2 \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx \, dt \right. \]

\[
- \frac{1}{2} \int_0^T \int_0^\ell \exp(-ct) \frac{\partial a}{\partial t} (\ell - x)^2 \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx \, dt
\]

\[
+ \frac{c}{2} \int_0^T \int_0^\ell \exp(-ct) (\ell - x)^2 \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx \, dt + \int_0^\ell \exp(-ct) a(x, \tau) \left| \frac{\partial u}{\partial t} (x, \tau) \right|^2 dx
\]

\[
- \int_0^T \int_0^\ell \exp(-ct) \frac{\partial a}{\partial t} \left| \frac{\partial u}{\partial t} \right|^2 dx \, dt + c \int_0^T \int_0^\ell \exp(-ct) a \left| \frac{\partial u}{\partial t} \right|^2 dx \, dt.
\] (2.15)
By using the elementary inequalities on the first integral in the left-hand side of (2.15), we obtain

\[
\frac{33}{2} \int_0^\ell \int_0 \exp(-ct)(\ell - x)^2 |\nabla u|^2 \, dx \, dt + \frac{3}{4} \int_0^\ell \int_0 \exp(-ct)(\ell - x)^2 \left| \frac{\partial^2 u}{\partial t^2} \right|^2 \, dx \, dt \\
+ \int_0^\ell a(x,0)|\psi|^2 \, dx + \frac{1}{2} \int_0^\ell a(x,0)(\ell - x)^2 \left| \frac{dfu}{dx} \right|^2 \, dx \\
\geq \int_0^\ell \int_0 \exp(-ct)(\ell - x)^2 \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \, dx \, dt - 2 \int_0^\ell \int_0 \exp(-ct) \left| \frac{\partial a}{\partial x} \right|^2 \left| \frac{\partial u}{\partial t} \right|^2 \, dx \, dt \\
+ \frac{1}{2} \int_0^\ell \exp(-ct)(\ell - x)^2 \left| \frac{\partial a}{\partial t} \right|^2 \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \, dx \\
+ \frac{1}{2} \int_0^\ell \exp(-ct)(\ell - x)^2 a \left| \frac{\partial a}{\partial x} \right|^2 \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 \, dx + \int_0^\ell \exp(-ct) a(x,\tau) \left| \frac{\partial u(x,\tau)}{\partial t} \right|^2 \, dx \\
- \int_0^\ell \int_0 \exp(-ct) \left( \frac{\partial a}{\partial t} \right)^2 \left| \frac{\partial u}{\partial t} \right|^2 \, dx \, dt + c \int_0^\ell \int_0 \exp(-ct) a \left| \frac{\partial u}{\partial t} \right|^2 \, dx \, dt.
\]

(2.16)

Now, from (1.1) we have

\[
\frac{1}{5} \int_0^\ell \int_0 \exp(-ct)(\ell - x)^2 |\nabla u|^2 \, dx \, dt \\
+ \frac{1}{5} \int_0^\ell \int_0 \exp(-ct)(\ell - x)^2 \left| \frac{\partial a}{\partial x} \right|^2 \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \, dx \, dt \\
+ \frac{1}{5} \int_0^\ell \int_0 \exp(-ct)(\ell - x)^2 \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \, dx \, dt \\
\geq \frac{1}{15} \int_0^\ell \int_0 \exp(-ct)(\ell - x)^2 \left( \frac{\partial^3 u}{\partial x^2 \partial t} \right)^2 \, dx \, dt.
\]

(2.17)

Combining inequalities (2.16), (2.17), and Lemmas 2.3 and 2.4, we get

\[
\frac{167}{10} \int_\Omega (\ell - x)^2 |\nabla u|^2 \, dx \, dt + \frac{a_1}{2} \int_0^\ell (\ell - x)^2 \left| \frac{dfu}{dx} \right|^2 \, dx \\
+ a_1 \int_0^\ell |\psi|^2 \, dx + \frac{1}{2} \int_0^\ell (\ell - x)^2 \left| \frac{dfu}{dx} \right|^2 \, dx + \int_0^\ell |\psi|^2 \, dx \\
\geq \exp(-ct) \left( \frac{1}{20} \int_0^\ell (\ell - x)^2 \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \, dx \, dt + \frac{1}{2} \int_0^\ell (\ell - x)^2 \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 \, dx \, dt \\
+ \int_0^\ell |u(x,\tau)|^2 \, dx + a_0 \int_0^\ell \left| \frac{\partial u}{\partial t} \right|^2 \, dx + \frac{1}{2} \int_0^\ell (\ell - x)^2 \left| \frac{\partial u}{\partial x} \right|^2 \, dx \\
+ a_0 \int_0^\ell \left( \frac{\partial a}{\partial t} \right)^2 \left| \frac{\partial u}{\partial t} \right|^2 \, dx \, dt \right).
\]

(2.18)

As the left-hand side of (2.18) is independent of \( \tau \), by replacing the right-hand side by its upper bound with respect to \( \tau \) in the interval \([0,T]\), we obtain the desired inequality. \( \square \)
3. Solvability of the problem. From estimates (2.1) and (2.3) it follows that the operator \( L : E \to F \) is continuous and its range is closed in \( F \). Therefore, the inverse operator \( L^{-1} \) exists and is continuous from the closed subspace \( R(L) \) onto \( E \), which means that \( L \) is a homomorphism from \( E \) onto \( R(L) \). To obtain the uniqueness of solution, it remains to show that \( R(L) = F \). The proof is based on the following lemma.

**Lemma 3.1.** Suppose that \( \partial^3 a / \partial x^2 \partial t \) is also bounded. Let \( D_0(L) = \{ u \in D(L) : l_1 u = 0, l_2 u = 0 \} \). If for \( u \in D_0(L) \) and some \( \omega \in L^2(\Omega) \),

\[
\int_{\Omega} (\ell - x) \partial^2 u \partial x \partial t \, dx \, dt = 0, \quad (3.1)
\]

then \( \omega = 0 \).

**Proof.** From (3.1) we have

\[
\int_{\Omega} (\ell - x) \frac{\partial^2 u}{\partial t^2} \partial x \partial t \, dx \, dt = \int_{\Omega} (\ell - x) \frac{\partial}{\partial x} \left( d \frac{\partial^2 u}{\partial x \partial t} \right) \partial x \partial t \, dx \, dt. \quad (3.2)
\]

If we introduce the smoothing operators with respect to \( t \) (see [13]) \( J^{-1}_\xi = (I + \xi \partial / \partial t)^{-1} \) and \( (J^{-1}_\xi)^* \), then these operators provide the solutions of the respective problems

\[
\xi \frac{d g_\xi(t)}{dt} + g_\xi(t) = g(t), \quad g_\xi(t)|_{t=0} = 0, \quad (3.3)
\]

\[
-\xi \frac{d g^*_\xi(t)}{dt} + g^*_\xi(t) = g(t), \quad g^*_\xi(t)|_{t=T} = 0, \quad (3.4)
\]

and also have the following properties: for any \( g \in L_2(0, T) \), the functions \( g_\xi = (J^{-1}_\xi) g \) and \( g^*_\xi = (J^{-1}_\xi)^* g \) are in \( W^1_2(0, T) \) such that \( g_\xi|_{t=0} = 0 \) and \( g^*_\xi|_{t=T} = 0 \). Moreover, \( J^{-1}_\xi \) commutes with \( \partial / \partial t \), so \( \int_0^T |g_\xi - g|^2 \, dt \to 0 \) and \( \int_0^T |g^*_\xi - g|^2 \, dt \to 0 \) for \( \xi \to 0 \).

Now, for given \( \omega(x, t) \), we introduce the function

\[
v(x, t) = \omega(x, t) - \int_0^x \frac{\omega(\xi, t)}{\ell - \xi} \, d\xi. \quad (3.5)
\]

Integrating by parts with respect to \( \xi \), we obtain

\[
\int_0^x v(\xi, t) \, d\xi = \int_0^x \omega(\xi, t) \, d\xi + \int_0^x \frac{\partial}{\partial \xi} (\ell - \xi) \int_0^\xi \frac{\omega(\eta, t)}{\ell - \eta} \, d\eta \, d\xi = (\ell - x)(\omega(x, t) - v(x, t)), \quad (3.6)
\]

which implies that

\[
(\ell - x)v + Jv = (\ell - x)u, \quad \int_0^\ell v(x, t) \, dx = 0. \quad (3.7)
\]

Then, from equality (3.2) we obtain

\[
-\int_{\Omega} \frac{\partial^2 u}{\partial t^2} Nv \, dx \, dt = \int_{\Omega} A(t) \frac{\partial u}{\partial t} v \, dx \, dt, \quad (3.8)
\]

where

\[
Nv = (\ell - x)v + Ju, \quad A(t)u = -\frac{\partial}{\partial x} \left( (\ell - x)a(x, t) \frac{\partial u}{\partial x} \right). \quad (3.9)
\]
Replace $\frac{\partial u}{\partial t}$ by the smoothed function $J^{-1}_\xi(\frac{\partial u}{\partial t})$ in (3.8) and use the relation

$$A(t)J^{-1}_\xi = J^{-1}_\xi A + \xi J^{-1}_\xi \frac{\partial A(\tau)}{\partial \tau} J^{-1}_\xi.$$  (3.10)

Then, by taking the adjoint of the operator $J^{-1}_\xi$, and by integrating by parts with respect to $t$ in the left-hand side, we obtain

$$\int_\Omega \frac{\partial u}{\partial t} N \frac{\partial v^*_\xi}{\partial t} \, dx \, dt = \int_\Omega A(t) \frac{\partial u}{\partial t} v^*_\xi \, dx \, dt + \xi \int_\Omega \frac{\partial A}{\partial t} \left( \frac{\partial u}{\partial t} \right)_\xi v^*_\xi \, dx \, dt.$$  (3.11)

The operator $A(t)$ has a continuous inverse on $L^2(0, \ell)$ defined by the relation

$$A^{-1}(t) g = - \int_0^\ell \frac{d\xi}{a(\xi, t)(\ell - \xi)} \int_0^\xi g(\eta) \, d\eta + c \int_0^\ell \frac{d\xi}{a(\xi, t)(\ell - \xi)},$$  (3.12)

where

$$c = \frac{\int_0^\ell (dx/a(x, t)) \int_0^\xi g(\xi) \, d\xi}{\int_0^\ell (dx/a(x, t))}, \quad \int_0^\ell A^{-1}(t) g \, dx = 0.$$  (3.13)

Hence, the function $(\frac{\partial u}{\partial t})_\xi$ can be represented in the form

$$(\frac{\partial u}{\partial t})_\xi = J^{-1}_\xi A^{-1}(t) A(t) \frac{\partial u}{\partial t}.$$  (3.14)

Then, $(\frac{\partial A}{\partial t})(\frac{\partial u}{\partial t})_\xi = A_\xi(t) A(t)(\frac{\partial u}{\partial t})$, where

$$A_\xi(t) = \left( \frac{\partial^2 a}{\partial x \partial t} J^{-1}_\xi - \frac{\partial a}{\partial t} J^{-1}_\xi \frac{1}{a} \frac{\partial a}{\partial x} \right) \frac{1}{a} \left( \int_0^\ell g(\eta, t) \, d\eta - c \right) + \frac{\partial a}{\partial t} J^{-1}_\xi \frac{1}{a} g,$$  (3.15)

where the constant $c$ is given by (3.13).

Consequently, equation (3.11) becomes

$$\int_\Omega \frac{\partial u}{\partial t} N \frac{\partial v^*_\xi}{\partial t} \, dx \, dt = \int_\Omega A(t) \frac{\partial u}{\partial t} (v^*_\xi + \xi A_\xi v^*_\xi) \, dx \, dt,$$  (3.16)

in which the conjugate operator $A^*_\xi(t)$ of $A_\xi(t)$ is defined by

$$A^*_\xi v^*_\xi = \frac{1}{a} (J^{-1}_\xi)^* \frac{\partial a}{\partial \tau} v^*_\xi + (Bv^*_\xi)(x) - (Bv^*_\xi)(0) \frac{\int_0^\ell (d\xi/a(\xi, t))}{\int_0^\ell (d\xi/a(\xi, t))},$$  (3.17)

where

$$(Bv^*_\xi)(x) = \int_x^\ell \frac{1}{a(\xi, t)} \left[ (J^{-1}_\xi)^* \frac{\partial^2 a}{\partial \xi \partial \tau} - \frac{1}{a(\xi, t)} \frac{\partial a}{\partial \xi} (J^{-1}_\xi)^* \frac{\partial a}{\partial \tau} \right] v^*_\xi(\xi, \tau) \, d\xi.$$  (3.18)
The left-hand side of (3.16) is a continuous linear functional of \( \partial u/\partial t \). Hence, the function \( h_\xi = v_\xi^* + xA_\xi^* v_\xi^* \) has the derivatives \((\ell - x)(\partial h_\xi/\partial x) \in L_2(\Omega)\), \((\partial/\partial x)((\ell - x)(\partial h_\xi/\partial x)) \in L_2(\Omega)\), and the following conditions are satisfied

\[
h_\xi |_{x=0} = 0, \ h_\xi |_{x=\ell} = 0, \ (\ell - x) \frac{\partial h_\xi}{\partial x} |_{x=\ell} = 0. \tag{3.19}
\]

From (3.17) we have

\[
(\ell - x) \frac{\partial h_\xi}{\partial x} = \left( I + \xi (J_\xi^{-1})^* \frac{\partial a}{\partial \tau} \right) \frac{\partial v_\xi^*}{\partial x}, \tag{3.20}
\]

\[
\frac{\partial}{\partial x} \left( (\ell - x) \frac{\partial h_\xi}{\partial x} \right) = \left( I + \xi (J_\xi^{-1})^* \frac{\partial a}{\partial \tau} \right) \frac{\partial}{\partial x} \left( (\ell - x) \frac{\partial v_\xi^*}{\partial x} \right)
+ \xi \left[ \frac{\partial a / \partial x}{a^2} (J_\xi^{-1})^* \frac{\partial a / \partial \tau}{a^2} \right] \frac{\partial^2 a}{\partial x \partial \tau} \left( x, t \right) \frac{\partial v_\xi^*}{\partial x}, \tag{3.21}
\]

\[
\left[ \left( I + \xi (J_\xi^{-1})^* \frac{\partial a}{\partial \tau} \right) v_\xi^* \right]_{x=0} = 0, \tag{3.22}
\]

\[
\left[ \left( I + \xi (J_\xi^{-1})^* \frac{\partial a}{\partial \tau} \right) v_\xi^* \right]_{x=\ell} = 0, \tag{3.23}
\]

\[
\left[ \left( I + \xi (J_\xi^{-1})^* \frac{\partial a}{\partial \tau} \right) (\ell - x) \frac{\partial v_\xi^*}{\partial x} \right]_{x=\ell} = 0. \tag{3.24}
\]

Since \( \|\xi (1/a)(J_\xi^{-1})^*(\partial a/\partial \tau)\|_{L_2(\Omega)} < 1 \) for sufficiently small \( \xi \), the operator \( I + \xi (1/a)(J_\xi^{-1})^*(\partial a/\partial \tau) \) has a continuous inverse on \( L_2(\Omega) \). In addition, the derivative of the above operator with respect to \( x \) is a bounded operator in \( L_2(\Omega) \). Therefore, from (3.20) and (3.21), the function \( v_\xi^* \) has derivatives \((\ell - x)(\partial^2 v_\xi^*/\partial x) \in L_2(\Omega)\) and \((\partial/\partial x)((\ell - x)(\partial v_\xi^*/\partial x)) \in L_2(\Omega)\).

In a similar way, we show that for each fixed \( x \in [0, \ell] \) and sufficiently small \( \xi \), the operator \( I + \xi (1/a)(J_\xi^{-1})^*(\partial a/\partial \tau) \) has a continuous inverse on \( L_2(0, T) \); hence, (3.22), and (3.23), and (3.24) imply that

\[
v_\xi^* |_{x=0} = 0, \ v_\xi^* |_{x=\ell} = 0, \ (\ell - x) \frac{\partial v_\xi^*}{\partial x} |_{x=\ell} = 0. \tag{3.25}
\]

So, for \( \xi \) sufficiently small, the function \( v_\xi^* \) has the same properties as \( h_\xi \). In addition, \( v_\xi^* \) satisfies the integral condition in (3.7).

Putting \( u = \int_0^t \int_\Omega \exp(c\eta) v_\xi^*(\eta, \tau) \, d\eta \, d\tau \) in (3.8), where the constant \( c \) satisfies \( ca_0 - a_3 - a_5^2/a_0 \geq 0 \), and using (3.4), we obtain

\[
\int_\Omega \exp(ct) v_\xi^* Nu \, dx \, dt = - \int_\Omega A(t) \frac{\partial u}{\partial t} \exp(-ct) \frac{\partial^2 u}{\partial t^2} \, dx \, dt + \xi \int_\Omega A(t) \frac{\partial u}{\partial t} \frac{\partial v_\xi^*}{\partial t} \, dx \, dt. \tag{3.26}
\]
Integrating by parts each term in the left-hand side of (3.26) and taking the real parts yield

\[
\Re \int_\Omega A(t) \frac{\partial u}{\partial t} \exp(-ct) \frac{\partial^2 u}{\partial x^2} \, dx \, dt \\
\geq c \int_\Omega (\ell - x) a(x,t) \exp(-ct) \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 \, dx \, dt \\
- \frac{1}{2} \int_\Omega (\ell - x) \frac{\partial a}{\partial t} \exp(-ct) \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 \, dx \, dt,
\]

(3.27)

Now, using (3.27) in (3.26) with the choice of \( c \) indicated above we have

\[
2 \Re \int_\Omega \exp(ct) v^\xi \tilde{N} \, dx \, dt \leq 0.
\]

(3.28)

Then, for \( \xi \to 0 \) we obtain

\[
2 \Re \int_\Omega \exp(ct) (\ell - x) |v|^2 \, dx \, dt + 2 \Re \int_\Omega \exp(ct) v J v \, dx \, dt \leq 0.
\]

(3.29)

Since \( \Re \int_\Omega \exp(ct) v J v \, dx \, dt = 0 \), we conclude that \( v = 0 \); hence, \( \omega = 0 \), which ends the proof of the lemma.

**Theorem 3.2.** The range \( R(L) \) of \( L \) coincides with \( F \).

**Proof.** Since \( F \) is a Hilbert space, we have \( R(L) = F \) if and only if the relation

\[
\int_\Omega (\ell - x)^2 \xi u f \, dx \, dt + \int_0^\ell \left[ (\ell - x)^2 \left( \frac{dl_1 u}{dx} \frac{d\varphi}{dx} + \frac{dl_2 u}{dx} \frac{d\psi}{dx} \right) + l_1 u \varphi + l_2 u \psi \right] \, dx = 0,
\]

(3.30)

for arbitrary \( u \in E \) and \( (f, \varphi, \psi) \in F \), implies that \( f = 0 \), \( \varphi = 0 \) and \( \psi = 0 \). Putting \( u \in D_0(L) \) in (3.30), we conclude from Lemma 3.1 that \( (\ell - x)f = 0 \). Hence,

\[
\int_0^\ell \left[ (\ell - x)^2 \left( \frac{dl_1 u}{dx} \frac{d\varphi}{dx} + \frac{dl_2 u}{dx} \frac{d\psi}{dx} \right) + l_1 u \varphi + l_2 u \psi \right] \, dx = 0 \quad \forall \, u \in D(L).
\]

(3.31)

Setting

\[
D_{0k}(L) = \left\{ u \in D(L) : u^{(k)} \big|_{t=0} = 0, \, k = 0, 1 \right\},
\]

(3.32)

and taking \( u \in D_{01}(L) \) in (3.31) yield

\[
\int_0^\ell (\ell - x)^2 \frac{dl_1 u}{dx} \frac{d\varphi}{dx} + l_1 u \varphi \, dx = 0.
\]

(3.33)

The range of the trace operator \( l_1 \) is everywhere dense in Hilbert space with the norm

\[
\left[ \int_0^\ell (\ell - x)^2 \left| \frac{d\varphi}{dx} \right|^2 + |\varphi|^2 \, dx \right]^{1/2}; \quad \text{hence,} \quad \varphi = 0.
\]

Likewise, for \( u \in D_{00}(L) \), we get \( \psi = 0 \).
REFERENCES


M. Denche: Institut de Mathématiques, Université Mentouri Constantine, Constantine 25000, Algeria

E-mail address: m_denche@hotmail.com

A. L. Marhoune: Institut de Mathématiques, Université Mentouri Constantine, Constantine 25000, Algeria
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Guest Editors

Edson Denis Leonel, Department of Statistics, Applied Mathematics and Computing, Institute of Geosciences and Exact Sciences, State University of São Paulo at Rio Claro, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob’evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru