ON n-FOLD FUZZY IMPLICATIVE/COMMUTATIVE IDEALS OF BCK-ALGEBRAS

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(Received 3 November 2000)

Abstract. We consider the fuzzification of the notion of an n-fold implicative ideal, an n-fold (weak) commutative ideal. We give characterizations of an n-fold fuzzy implicative ideal. We establish an extension property for n-fold fuzzy commutative ideals.

2000 Mathematics Subject Classification. 06F35, 03G25, 03E72.

1. Introduction. Huang and Chen [1] introduced the notion of n-fold implicative ideals and n-fold (weak) commutative ideals. The aim of this paper is to discuss the fuzzification of n-fold implicative ideals, n-fold commutative ideals and n-fold weak commutative ideals. We show that every n-fold fuzzy implicative ideal is an n-fold fuzzy positive implicative ideal, and so a fuzzy ideal, and give a condition for a fuzzy ideal to be an n-fold fuzzy implicative ideal. Using the level set, we provide a characterization of an n-fold fuzzy implicative ideal. We also give a condition for a fuzzy ideal to be an n-fold fuzzy (weak) commutative ideal. We show that every n-fold fuzzy positive implicative ideal which is an n-fold fuzzy weak commutative ideal is an n-fold fuzzy implicative ideal. Finally, we establish an extension property for n-fold fuzzy commutative ideals.

2. Preliminaries. We include some elementary aspects of BCK-algebras that are necessary for this paper, and for more details we refer to [1, 2, 4, 5]. By a BCK-algebra we mean an algebra \((X; *, 0)\) of type \((2, 0)\) satisfying the axioms:

(I) \(((x * y) * (x * z)) * (z * y) = 0\),
(II) \((x * (x * y)) * y = 0\),
(III) \(x * x = 0\),
(IV) \(0 * x = 0\),
(V) \(x * y = 0\) and \(y * x = 0\) imply \(x = y\), for all \(x, y, z \in X\).

We can define a partial ordering \(\leq\) on \(X\) by \(x \leq y\) if and only if \(x * y = 0\). In any BCK-algebra \(X\), the following hold:

(P1) \(x * 0 = x\),
(P2) \(x * y \leq x\),
(P3) \((x * y) * z = (x * z) * y\),
(P4) \((x * z) * (y * z) \leq x * y\),
(P5) \(x \leq y\) implies \(x * z \leq y * z\) and \(z * y \leq z * x\).

Throughout, \(X\) will always mean a BCK-algebra unless otherwise specified. A non-empty subset \(I\) of \(X\) is called an ideal of \(X\) if it satisfies:

(I1) \(0 \in I\),
(I2) $x \ast y \in I$ and $y \in I$ imply $x \in I$.
A nonempty subset $I$ of $X$ is said to be an implicative ideal of $X$ if it satisfies:
(I1) $0 \in I$,
(I3) $(x \ast (y \ast x)) \ast z \in I$ and $z \in I$ imply $x \in I$.
A nonempty subset $I$ of $X$ is said to be a commutative ideal of $X$ if it satisfies:
(I1) $0 \in I$,
(I4) $(x \ast y) \ast z \in I$ and $z \in I$ imply $x \ast (y \ast (y \ast x)) \in I$.
We now review some fuzzy logic concepts. A fuzzy set in a set $X$ is a function
$f : X \to [0, 1]$. For a fuzzy set $f$ in $X$ and $t \in [0, 1]$ define $U(f; t)$ to be the set
$U(f; t) := \{x \in X | f(x) \geq t\}$.
A fuzzy set $f$ in $X$ is said to be a fuzzy implicative ideal of $X$ if it satisfies:
(F1) $f(0) \geq f(x)$ for all $x \in X$,
(F3) $f(x) \geq \min\{f((x \ast (y \ast x)) \ast z), f(z)\}$ for all $x, y, z \in X$.
A fuzzy set $f$ in $X$ is called a fuzzy commutative ideal of $X$ if it satisfies:
(F1) $f(0) \geq f(x)$ for all $x \in X$,
(F4) $f(x \ast (y \ast (y \ast x))) \geq \min\{f((x \ast y) \ast z), f(z)\}$ for all $x, y, z \in X$.

3. $n$-fold fuzzy implicative ideals. For any elements $x$ and $y$ of a BCK-algebra $X$, $x \ast y^n$ denotes
$(\cdots ((x \ast y) \ast y) \ast \cdots) \ast y$ (3.1)
in which $y$ occurs $n$ times. Huang and Chen [1] introduced the concept of $n$-fold
implicative ideals as follows.

**Definition 3.1** (see [1]). A subset $A$ of $X$ is called an $n$-fold implicative ideal of $X$ if
(I1) $0 \in A$,
(I5) $(x \ast (y \ast x^n)) \ast z \in A$ and $z \in A$ imply $x \in A$ for every $x, y, z \in X$.
We consider the fuzzification of the concept of $n$-fold implicative ideal.

**Definition 3.2.** A fuzzy set $f$ in $X$ is called an $n$-fold fuzzy implicative ideal of $X$ if
(F1) $f(0) \geq f(x)$ for all $x \in X$,
(F5) $f(x) \geq \min\{f((x \ast (y \ast x^n)) \ast z), f(z)\}$ for every $x, y, z \in X$.
Notice that the 1-fold fuzzy implicative ideal is a fuzzy implicative ideal.

**Theorem 3.3.** Every $n$-fold fuzzy implicative ideal is a fuzzy ideal.

**Proof.** The condition (F2) follows from taking $y = 0$ in (F5).

The following example shows that the converse of Theorem 3.3 may not be true.
### Example 3.4

Let \( X = \mathbb{N} \cup \{0\} \), where \( \mathbb{N} \) is the set of natural numbers, in which the operation \( \ast \) is defined by \( x \ast y = \max\{0, x - y\} \) for all \( x, y \in X \). Then \( X \) is a BCK-algebra (see [1, Example 1.3]). Let \( \mu \) be a fuzzy set in \( X \) given by \( \mu(0) = t_0 > t_1 = \mu(x) \) for all \( x \neq 0 \in X \). Then \( \mu \) is a fuzzy ideal of \( X \). But \( \mu \) is not a 2-fold fuzzy implicative ideal of \( X \) because

\[
\mu(3) = t_1 < t_0 = \mu(0) = \min\{\mu((3 \ast (14 \ast 3^2)) \ast 0), \mu(0)\}. \tag{3.2}
\]

We give a condition for a fuzzy ideal to be an \( n \)-fold fuzzy implicative ideal.

### Theorem 3.5

A fuzzy ideal \( \mu \) of \( X \) is \( n \)-fold fuzzy implicative if and only if \( \mu(x) \geq \mu(x \ast (y \ast x^n)) \) for all \( x, y \in X \).

**Proof.** Necessity is by taking \( z = 0 \) in (F5). Suppose that a fuzzy ideal \( \mu \) satisfies the inequality \( \mu(x) \geq \mu(x \ast (y \ast x^n)) \) for all \( x, y \in X \). Then

\[
\mu(x) \geq \mu(x \ast (y \ast x^n)) \geq \min\{\mu((x \ast (y \ast x^n)) \ast z), \mu(z)\}. \tag{3.3}
\]

Hence \( \mu \) is an \( n \)-fold fuzzy implicative ideal of \( X \).

### Theorem 3.6

A fuzzy set \( \mu \) in \( X \) is an \( n \)-fold fuzzy implicative ideal of \( X \) if and only if the nonempty level set \( U(\mu; t) \) of \( \mu \) is an \( n \)-fold implicative ideal of \( X \) for every \( t \in [0, 1] \).

**Proof.** Assume that \( \mu \) is an \( n \)-fold fuzzy implicative ideal of \( X \) and \( U(\mu; t) \neq \emptyset \) for every \( t \in [0, 1] \). Then there exists \( x \in U(\mu; t) \). It follows from (F1) that \( \mu(0) \geq \mu(x) \geq t \) so that \( 0 \in U(\mu; t) \). Let \( x, y, z \in X \) be such that \( (x \ast (y \ast x^n)) \ast z \in U(\mu; t) \) and \( z \in U(\mu; t) \). Then \( \mu((x \ast (y \ast x^n)) \ast z) \geq t \) and \( \mu(z) \geq t \), which imply from (F5) that

\[
\mu(x) \geq \min\{\mu((x \ast (y \ast x^n)) \ast z), \mu(z)\} \geq t \tag{3.4}
\]

so that \( x \in U(\mu; t) \). Therefore \( U(\mu; t) \) is an \( n \)-fold implicative ideal of \( X \). Conversely, suppose that \( U(\mu; t) \neq \emptyset \) is an \( n \)-fold implicative ideal of \( X \) for every \( t \in [0, 1] \). For any \( x \in X \), let \( \mu(x) = t \). Then \( x \in U(\mu; t) \). Since \( 0 \in U(\mu; t) \), we get \( \mu(0) = t = \mu(x) \) and so \( \mu(0) \geq \mu(x) \) for all \( x \in X \). Now assume that there exist \( a, b, c \in X \) such that

\[
\mu(a) < \min\{\mu((a \ast (b \ast a^n)) \ast c), \mu(c)\}. \tag{3.5}
\]

Selecting \( s_0 = (1/2)\mu(a) + \min\{\mu((a \ast (b \ast a^n)) \ast c), \mu(c)\} \), then

\[
\mu(a) < s_0 < \min\{\mu((a \ast (b \ast a^n)) \ast c), \mu(c)\}. \tag{3.6}
\]

It follows that \( (a \ast (b \ast a^n)) \ast c \in U(\mu; s_0) \), \( c \in U(\mu; s_0) \), and \( a \notin U(\mu; s_0) \). This is a contradiction. Hence \( \mu \) is an \( n \)-fold fuzzy implicative ideal of \( X \).

### Definition 3.7

(see [3]). A fuzzy set \( \mu \) in \( X \) is called an \( n \)-fold fuzzy positive implicative ideal of \( X \) if

\((F1) \quad \mu(0) \geq \mu(x) \) for all \( x \in X \),
\((F6) \quad \mu(x \ast y^n) \geq \min\{\mu((x \ast y^{n+1}) \ast z), \mu(z)\} \) for all \( x, y, z \in X \).
**Lemma 3.8** (see [3, Theorem 3.13]). Let $\mu$ be a fuzzy set in $X$. Then $\mu$ is an $n$-fold fuzzy positive implicative ideal of $X$ if and only if the nonempty level set $U(\mu; t)$ of $\mu$ is an $n$-fold positive implicative ideal of $X$ for every $t \in [0, 1]$.

**Lemma 3.9** (see [1, Theorem 2.5]). Every $n$-fold implicative ideal is an $n$-fold positive implicative ideal.

Using Theorem 3.6 and Lemmas 3.8 and 3.9, we have the following theorem.

**Theorem 3.10.** Every $n$-fold fuzzy implicative ideal is an $n$-fold fuzzy positive implicative ideal.

4. $n$-fold fuzzy commutative ideals

**Definition 4.1** (see [1]). A subset $A$ of $X$ is called an $n$-fold commutative ideal of $X$ if

(I1) $0 \in A$,

(I6) $(x \ast y) \ast z \in A$ and $z \in A$ imply $x \ast (y \ast (y \ast x^n)) \in A$ for all $x, y, z \in X$.

A subset $A$ of $X$ is called an $n$-fold weak commutative ideal of $X$ if

(I1) $0 \in A$,

(I7) $(x \ast (x \ast y^n)) \ast z \in A$ and $z \in A$ imply $y \ast (y \ast x) \in A$ for all $x, y, z \in X$.

We consider the fuzzification of $n$-fold (weak) commutative ideals as follows.

**Definition 4.2.** A fuzzy set $\mu$ in $X$ is called an $n$-fold fuzzy commutative ideal of $X$ if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F7) $\mu(x \ast (y \ast (y \ast x^n))) \geq \min\{\mu((x \ast y) \ast z), \mu(z)\}$ for all $x, y, z \in X$.

A fuzzy set $\mu$ in $X$ is called an $n$-fold fuzzy weak commutative ideal of $X$ if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F8) $\mu(y \ast (y \ast x)) \geq \min\{\mu((x \ast (x \ast y^n)) \ast z), \mu(z)\}$ for all $x, y, z \in X$.

Note that the 1-fold fuzzy commutative ideal is a fuzzy commutative ideal. Putting $y = 0$ and $y = x$ in (F7) and (F8), respectively, we know that every $n$-fold fuzzy commutative (or fuzzy weak commutative) ideal is a fuzzy ideal.

**Theorem 4.3.** Let $\mu$ be a fuzzy ideal of $X$. Then

(i) $\mu$ is an $n$-fold fuzzy commutative ideal of $X$ if and only if

$$\mu(x \ast (y \ast (y \ast x^n))) \geq \mu(x \ast y) \quad \forall x, y \in X.$$  \hspace{1cm} (4.1)

(ii) $\mu$ is an $n$-fold fuzzy weak commutative ideal of $X$ if and only if

$$\mu(y \ast (y \ast x)) \geq \mu(x \ast (x \ast y^n)) \quad \forall x, y \in X.$$  \hspace{1cm} (4.2)

**Proof.** The proof is straightforward. \qed

**Lemma 4.4** (see [3, Theorem 3.12]). A fuzzy set $\mu$ in $X$ is an $n$-fold fuzzy positive implicative ideal of $X$ if and only if $\mu$ is a fuzzy ideal of $X$ in which the following inequality holds:

(F9) $\mu((x \ast z^n) \ast (y \ast z^n)) \geq \mu((x \ast y) \ast z^n) \quad \forall x, y, z \in X$. 
\textbf{Theorem 4.5.} If $\mu$ is both an $n$-fold fuzzy positive implicative ideal and an $n$-fold fuzzy weak commutative ideal of $X$, then it is an $n$-fold fuzzy implicative ideal of $X$.

\textbf{Proof.} Let $x, y \in X$. Using \textbf{Theorem 4.3(ii)}, \textbf{Lemma 4.4}, (P3), and (III), we have
\begin{align*}
\mu(x \ast (x \ast (y \ast x^n))) & \geq \mu((y \ast x^n) \ast ((y \ast x^n) \ast x^n)) \\
& \geq \mu((y \ast (y \ast x^n)) \ast x^n) \\
& = \mu((y \ast x^n) \ast (y \ast x^n)) \\
& = \mu(0). \\
\end{align*}
(4.3)

It follows from (F1) and (F2) that
\begin{align*}
\mu(x) & \geq \min\{\mu(x \ast (x \ast (y \ast x^n))), \mu(x \ast (y \ast x^n))\} \\
& \geq \min\{\mu(0), \mu(x \ast (y \ast x^n))\} \\
& = \mu(x \ast (y \ast x^n)). \\
\end{align*}
(4.4)

so from \textbf{Theorem 3.5}, $\mu$ is an $n$-fold fuzzy implicative ideal of $X$. \hfill \Box

\textbf{Theorem 4.6} (extension property for $n$-fold fuzzy commutative ideals). Let $\mu$ and $\nu$ be fuzzy ideals of $X$ such that $\mu(0) = \nu(0)$ and $\mu \subseteq \nu$, that is, $\mu(x) \leq \nu(x)$ for all $x \in X$. If $\mu$ is an $n$-fold fuzzy commutative ideal of $X$, then so is $\nu$.

\textbf{Proof.} Let $x, y \in X$. Taking $u = x \ast (x \ast y)$, we have
\begin{align*}
\nu(0) = \mu(0) = \mu(u \ast y) \\
& \leq \mu(u \ast (y \ast (y \ast u^n))) \\
& \leq \nu(u \ast (y \ast (y \ast u^n))) \\
& = \nu((x \ast (x \ast y)) \ast (y \ast (y \ast u^n))) \\
& = \nu((x \ast (y \ast (y \ast u^n))) \ast (x \ast y)). \\
\end{align*}
(4.5)

Since $x \ast (y \ast (y \ast x^n)) \leq x \ast (y \ast (y \ast u^n))$ and since $\nu$ is order reversing, it follows that
\begin{align*}
\nu(x \ast (y \ast (y \ast x^n))) & \geq \nu(x \ast (y \ast (y \ast u^n))) \\
& \geq \min\{\nu((x \ast (y \ast (y \ast u^n))) \ast (x \ast y)), \nu(x \ast y)\} \\
& \geq \min\{\nu(0), \nu(x \ast y)\} \\
& = \nu(x \ast y). \\
\end{align*}
(4.6)

Hence, by \textbf{Theorem 4.3(i)}, $\nu$ is an $n$-fold fuzzy commutative ideal of $X$. \hfill \Box

\textbf{Acknowledgement.} This work was supported by Korea Research Foundation Grant (KRF-2000-005-D00003).

\textbf{References}


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