ON $\theta$-PRECONTINUOUS FUNCTIONS

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ABSTRACT. We introduce a new class of functions called $\theta$-precontinuous functions which is contained in the class of weakly precontinuous (or almost weakly continuous) functions and contains the class of almost precontinuous functions. It is shown that the $\theta$-precontinuous image of a $p$-closed space is quasi $H$-closed.

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1. Introduction. A subset $A$ of a topological space $X$ is said to be preopen [14] or nearly open [26] if $A \subset \text{Int}(\text{Cl}(A))$. A function $f : X \to Y$ is called precontinuous [14] if the preimage $f^{-1}(V)$ of each open set $V$ of $Y$ is preopen in $X$. Precontinuity was called near continuity by Pták [26] and also called almost continuity by Frolík [9] and Husain [10]. In 1985, Janković [12] introduced almost weak continuity as a weak form of precontinuity. Popa and Noiri [23] introduced weak precontinuity and showed that almost weak continuity is equivalent to weak precontinuity. Paul and Bhattacharyya [21] called weakly precontinuous functions quasi precontinuous and obtained the further properties of quasi precontinuity. Recently, Nasef and Noiri [16] have introduced and investigated the notion of almost precontinuity. Quite recently, Jafari and Noiri [11] investigated the further properties of almost precontinuous functions.

In this paper, we introduce a new class of functions called $\theta$-precontinuous functions which is contained in the class of weakly precontinuous functions and contains the class of almost precontinuous functions. We obtain basic properties of $\theta$-precontinuous functions. It is shown in the last section that the $\theta$-precontinuous images of $p$-closed (resp., $\beta$-connected) spaces are quasi $H$-closed (resp., semi-connected).

2. Preliminaries. Throughout, by $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) we denote topological spaces. Let $S$ be a subset of $X$. We denote the interior and the closure of $S$ by $\text{Int}(S)$ and $\text{Cl}(S)$, respectively. A subset $S$ is said to be preopen [14] (resp., semi-open [13], $\alpha$-open [17]) if $S \subset \text{Int}(\text{Cl}(S))$ (resp., $S \subset \text{Cl}(\text{Int}(S))$, $S \subset \text{Int}(\text{Cl}(\text{Int}(S)))$). The complement of a preopen set is called preclosed. The intersection of all preclosed sets containing $S$ is called the preclosure of $S$ and is denoted by $p\text{Cl}(S)$. The preinterior of $S$ is defined by the union of all preopen sets contained in $S$ and is denoted by $p\text{Int}(S)$. The family of all preopen sets of $X$ is denoted by $PO(X)$. We set $PO(X, x) = \{U : x \in U$ and $U \in PO(X)\}$. A point $x$ of $X$ is called a $\theta$-cluster point of $S$ if $\text{Cl}(U) \cap S \neq \emptyset$ for every open set $U$ of $X$ containing $x$. The set of all $\theta$-cluster points of $S$ is called the $\theta$-closure of $S$ and is denoted by $\text{Clo}_\theta(S)$. A subset $S$ is said to be $\theta$-closed [27] if $S = \text{Cl}_\theta(S)$. The complement of a $\theta$-closed set is said to be $\theta$-open. A point $x$ of $X$
is called a pre $\theta$-cluster point of $S$ if $pCl(U) \cap S \neq \emptyset$ for every preopen set $U$ of $X$ containing $x$. The set of all pre $\theta$-cluster points of $S$ is called the pre $\theta$-closure of $S$ and is denoted by $pCl_\theta(S)$. A subset $S$ is said to be pre $\theta$-closed [20] if $S = pCl_\theta(S)$.

The complement of a pre $\theta$-closed set is said to be pre $\theta$-open.

**Definition 2.1.** A function $f : X \to Y$ is said to be precontinuous [14] (resp., almost precontinuous [16], weakly precontinuous [23] or quasi precontinuous [21]) if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in PO(X,x)$ such that $f(U) \subset V$ (resp., $f(U) \subset Int(Cl(V))$, $f(U) \subset Cl(V)$).

**Definition 2.2.** A function $f : X \to Y$ is said to be almost weakly continuous [12] if $f^{-1}(V) \subset Int(Cl(f^{-1}(Cl(V))))$ for every open set $V$ of $Y$.

**Definition 2.3.** A function $f : X \to Y$ is said to be strongly $\theta$-precontinuous [19] if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in PO(X,x)$ such that $f(pCl(U)) \subset V$.

**Definition 2.4.** A function $f : X \to Y$ is said to be $\theta$-precontinuous if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in PO(X,x)$ such that $f(pCl(U)) \subset Cl(V)$.

**Remark 2.5.** By the above definitions and Theorem 3.3 below, we have the following implications and none of these implications is reversible by [19, Example 2.2], [11, Example 2.9], and Examples 2.6 and 5.11 below.

\[
\text{strongly } \theta\text{-precontinuous} \Rightarrow \text{precontinuous} \Rightarrow \text{almost precontinuous} \\
\Rightarrow \theta\text{-precontinuous} \Rightarrow \text{weakly precontinuous.}
\] (2.1)

**Example 2.6.** This example is due to Arya and Deb [4]. Let $X$ be the set of all real numbers. The topology $\tau$ on $X$ is the cocountable topology. Let $Y = \{a,b,c\}$, $\sigma = \{\emptyset,Y,\{a\},\{c\},\{a,c\}\}$. We define a function $f : (X,\tau) \to (Y,\sigma)$ by $f(x) = a$ if $x$ is rational; $f(x) = b$ if $x$ is irrational. Then $f$ is a $\theta$-precontinuous function which is not almost precontinuous.

**3. Characterizations**

**Theorem 3.1.** For a function $f : X \to Y$ the following properties are equivalent:

(1) $f$ is $\theta$-precontinuous;

(2) $pCl_\theta(f^{-1}(B)) \subset f^{-1}(Cl_\theta(B))$ for every subset $B$ of $Y$;

(3) $f(pCl_\theta(A)) \subset Cl_\theta(f(A))$ for every subset $A$ of $X$.

**Proof.** (1)$\Rightarrow$(2). Let $B$ be any subset of $Y$. Suppose that $x \notin f^{-1}(Cl_\theta(B))$. Then $f(x) \notin Cl_\theta(B)$ and there exists an open set $V$ containing $f(x)$ such that $Cl(V) \cap B = \emptyset$. Since $f$ is $\theta$.p.c., there exists $U \in PO(X,x)$ such that $f(pCl(U)) \subset Cl(V)$. Therefore, we have $f(pCl(U)) \cap B = \emptyset$ and $pCl(U) \cap f^{-1}(B) = \emptyset$. This shows that $x \notin pCl_\theta(f^{-1}(B))$. Thus, we obtain $pCl_\theta(f^{-1}(B)) \subset f^{-1}(Cl_\theta(B))$. 


(2)⇒(3). Let A be any subset of X. Then we have \( \text{pCl}_\theta(A) \subset \text{pCl}_\theta(f^{-1}(f(A))) \subset f^{-1}((\text{Cl}_\theta(f(A)))) \) and hence \( f(\text{pCl}_\theta(A)) \subset \text{Cl}_\theta(f(A)). \)

(3)⇒(2). Let B be a subset of Y. We have \( f(\text{pCl}_\theta(f^{-1}(B))) \subset \text{Cl}_\theta(f(f^{-1}(B))) \subset \text{Cl}_\theta(B) \) and hence \( \text{pCl}_\theta(f^{-1}(B)) \subset f^{-1}(\text{Cl}_\theta(B)). \)

(2)⇒(1). Let \( x \in X \) and \( V \) be an open set of \( Y \) containing \( f(x) \). Then we have \( \text{Cl}(V) \cap (Y - \text{Cl}(V)) = \emptyset \) and \( f(x) \notin \text{Cl}_\theta(Y - \text{Cl}(V)). \) Hence, \( x \notin f^{-1}(\text{Cl}_\theta(Y - \text{Cl}(V))) \) and \( x \notin \text{pCl}_\theta(f^{-1}(Y - \text{Cl}(V))). \) There exists \( U \in \text{PO}(X,x) \) such that \( \text{pCl}(U) \cap f^{-1}(Y - \text{Cl}(V)) = \emptyset \); hence \( f(\text{pCl}(U)) \subset \text{Cl}(V). \) Therefore, \( f \) is \( \theta.p.c. \).

**Theorem 3.2.** For a function \( f : X \to Y \) the following properties are equivalent:

1. \( f \) is \( \theta \)-precontinuous;
2. \( f^{-1}(V) \subset \text{pInt}_\theta(f^{-1}(\text{Cl}(V))) \) for every open set \( V \) of \( Y \);
3. \( \text{pCl}_\theta(f^{-1}(V)) \subset f^{-1}(\text{Cl}(V)) \) for every open set \( V \) of \( Y \).

**Proof.** (1)⇒(2). Suppose that \( V \) is any open set of \( Y \) and \( x \in f^{-1}(V) \). Then \( f(x) \in V \) and there exists \( U \in \text{PO}(X,x) \) such that \( f(\text{pCl}(U)) \subset \text{Cl}(V). \) Therefore, \( x \in U \subset \text{pCl}(U) \subset f^{-1}(\text{Cl}(V)). \) This shows that \( x \in \text{pInt}_\theta(f^{-1}(\text{Cl}(V))). \) Therefore, we obtain \( f^{-1}(V) \subset \text{pInt}_\theta(f^{-1}(\text{Cl}(V))). \)

(2)⇒(3). Suppose that \( V \) is any open set of \( Y \) and \( x \notin f^{-1}(\text{Cl}(V)). \) Then \( f(x) \notin \text{Cl}(V) \) and there exists an open set \( W \) containing \( f(x) \) such that \( W \cap V = \emptyset; \) hence \( \text{Cl}(W) \cap V = \emptyset. \) Therefore, we have \( f^{-1}(\text{Cl}(W)) \cap f^{-1}(V) = \emptyset. \) Since \( x \notin f^{-1}(W), \) by (2) \( x \in \text{pInt}_\theta(f^{-1}(\text{Cl}(W))). \) There exists \( U \in \text{PO}(X,x) \) such that \( \text{pCl}(U) \subset f^{-1}(\text{Cl}(W)). \) Thus we have \( \text{pCl}(U) \cap f^{-1}(V) = \emptyset \) and hence \( x \notin \text{pCl}_\theta(f^{-1}(V)). \) This shows that \( \text{pCl}_\theta(f^{-1}(V)) \subset f^{-1}(\text{Cl}(V)). \)

(3)⇒(1). Suppose that \( x \in X \) and \( V \) is any open set of \( Y \) containing \( f(x) \). Then \( V \cap (Y - \text{Cl}(V)) = \emptyset \) and \( f(x) \notin \text{Cl}(Y - \text{Cl}(V)). \) Therefore, \( x \notin f^{-1}(\text{Cl}(Y - \text{Cl}(V))) \) and \( x \notin \text{pCl}_\theta(f^{-1}(Y - \text{Cl}(V))). \) There exists \( U \in \text{PO}(X,x) \) such that \( \text{pCl}(U) \cap f^{-1}(Y - \text{Cl}(V)) = \emptyset. \) Therefore, we obtain \( f(\text{pCl}(U)) \subset \text{Cl}(V). \) This shows that \( f \) is \( \theta.p.c. \).

**Theorem 3.3.** For a function \( f : X \to Y \) the following properties hold:

1. if \( f \) is almost precontinuous, then it is \( \theta \)-precontinuous;
2. if \( f \) is \( \theta \)-precontinuous, then it is weakly precontinuous.

**Proof.** Statement (2) is obvious. We will show statement (1). Suppose that \( x \in X \) and \( V \) is any open set of \( Y \) containing \( f(x) \). Since \( f \) is almost precontinuous, \( f^{-1}(\text{Int}(\text{Cl}(V))) \) is preopen and \( f^{-1}(\text{Cl}(V)) \) is preclosed in \( X \) by [16, Theorem 3.1]. Now, set \( U = f^{-1}(\text{Int}(\text{Cl}(V))) \). Then we have \( U \in \text{PO}(X,x) \) and \( \text{pCl}(U) \subset f^{-1}(\text{Cl}(V)). \) Therefore, we obtain \( f(\text{pCl}(U)) \subset \text{Cl}(V). \) This shows that \( f \) is \( \theta.p.c. \).

**Corollary 3.4.** Let \( Y \) be a regular space. Then, for a function \( f : X \to Y \) the following properties are equivalent:

1. \( f \) is strongly \( \theta \)-precontinuous;
2. \( f \) is precontinuous;
3. \( f \) is almost precontinuous;
4. \( f \) is \( \theta \)-precontinuous;
5. \( f \) is weakly precontinuous.

**Proof.** This is an immediate consequence of [19, Theorem 3.2].
**Definition 3.5.** A topological space $X$ is said to be pre-regular [20] if for each preclosed set $F$ and each point $x \in X - F$, there exist disjoint preopen sets $U$ and $V$ such that $x \in U$ and $F \subset V$.

**Lemma 3.6** (see [20]). A topological space $X$ is pre-regular if and only if for each $U \in \text{PO}(X)$ and each point $x \in U$, there exists $V \in \text{PO}(X,x)$ such that $x \in V \subset p\text{Cl}(V) \subset U$.

**Theorem 3.7.** Let $X$ be a pre-regular space. Then $f : X \to Y$ is $\theta$.p.c. if and only if it is weakly precontinuous.

**Proof.** Suppose that $f$ is weakly precontinuous. Let $x \in X$ and $V$ be any open set of $Y$ containing $f(x)$. Then, there exists $U \in \text{PO}(X,x)$ such that $f(U) \subset \text{Cl}(V)$. Since $X$ is pre-regular, there exists $U_s \in \text{PO}(X,x)$ such that $x \in U_s \subset p\text{Cl}(U_s) \subset U$. Therefore, we obtain $f(p\text{Cl}(U_s)) \subset \text{Cl}(V)$. This shows that $f$ is $\theta$.p.c. \hfill $\Box$

**Theorem 3.8.** Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of $f$ defined by $g(x) = (x,f(x))$ for each $x \in X$. Then $g$ is $\theta$.p.c. if and only if $f$ is $\theta$.p.c.

**Proof**

**Necessity.** Suppose that $g$ is $\theta$.p.c. Let $x \in X$ and $V$ be an open set of $Y$ containing $f(x)$. Then $X \times V$ is an open set of $X \times Y$ containing $g(x)$. Since $g$ is $\theta$.p.c., there exists $U \in \text{PO}(X,x)$ such that $g(p\text{Cl}(U)) \subset \text{Cl}(X \times V)$. It follows that $\text{Cl}(X \times V) = X \times \text{Cl}(V)$. Therefore, we obtain $f(p\text{Cl}(U)) \subset \text{Cl}(V)$. This shows that $f$ is $\theta$.p.c.

**Sufficiency.** Let $x \in X$ and $W$ be any open set of $X \times Y$ containing $g(x)$. There exist open sets $U_1 \subset X$ and $V \subset Y$ such that $g(x) = (x,f(x)) \in U_1 \times V \subset W$. Since $f$ is $\theta$.p.c., there exists $U_2 \in \text{PO}(X,x)$ such that $f(p\text{Cl}(U_2)) \subset \text{Cl}(V)$. Let $U = U_1 \cap U_2$, then $U \in \text{PO}(X,x)$. Therefore, we obtain $g(p\text{Cl}(U)) \subset \text{Cl}(U_1) \times f(p\text{Cl}(U_2)) \subset \text{Cl}(U_1) \times \text{Cl}(V) \subset \text{Cl}(W)$. This shows that $g$ is $\theta$.p.c. \hfill $\Box$

4. Some properties

**Lemma 4.1** (see [15]). Let $A$ and $X_0$ be subsets of a space $X$.

1. If $A \in \text{PO}(X)$ and $X_0$ is semi-open in $X$, then $A \cap X_0 \in \text{PO}(X_0)$.
2. If $A \in \text{PO}(X_0)$ and $X_0 \in \text{PO}(X)$, then $A \in \text{PO}(X)$.

**Lemma 4.2** (see [7]). Let $A$ and $X_0$ be subsets of a space $X$ such that $A \subset X_0 \subset X$. Let $p\text{Cl}_{X_0}(A)$ denote the preclosure of $A$ in the subspace $X_0$.

1. If $X_0$ is semi-open in $X$, then $p\text{Cl}_{X_0}(A) \subset p\text{Cl}(A)$.
2. If $A \subset \text{PO}(X_0)$ and $X_0 \in \text{PO}(X)$, then $p\text{Cl}(A) \subset p\text{Cl}_{X_0}(A)$.

**Theorem 4.3.** If $f : X \to Y$ is $\theta$.p.c. and $X_0$ is a semi-open subset of $X$, then the restriction $f / X_0 : X_0 \to Y$ is $\theta$.p.c.

**Proof.** For any $x \in X_0$ and any open neighborhood $V$ of $f(x)$, there exists $U \in \text{PO}(X,x)$ such that $f(p\text{Cl}(U)) \subset \text{Cl}(V)$ since $f$ is $\theta$.p.c. Put $U_0 = U \cap X_0$, then by Lemmas 4.1 and 4.2 $U_0 \in \text{PO}(X_0,x)$ and $p\text{Cl}_{X_0}(U_0) \subset p\text{Cl}(U_0)$. Therefore, we obtain

$$
(f / X_0)(p\text{Cl}_{X_0}(U_0)) = f(p\text{Cl}_{X_0}(U_0)) \subset f(p\text{Cl}(U_0)) \subset f(p\text{Cl}(U)) \subset \text{Cl}(V). \quad (4.1)
$$

This shows that $f / X_0$ is $\theta$.p.c. \hfill $\Box$
**Theorem 4.4.** A function \( f : X \rightarrow Y \) is \( \theta \)-p.c. if for each \( x \in X \) there exists \( X_0 \in \text{PO}(X, x) \) such that the restriction \( f/X_0 : X_0 \rightarrow Y \) is \( \theta \)-p.c.

**Proof.** Let \( x \in X \) and \( V \) be any open neighborhood of \( f(x) \). There exists \( X_0 \in \text{PO}(X, x) \) such that \( f/X_0 : X_0 \rightarrow Y \) is \( \theta \)-p.c. Thus, there exists \( U \in \text{PO}(X, x) \) such that \( (f/X_0)(\text{pCl}(X_0(U))) \subseteq \text{Cl}(V) \). By Lemmas 4.1 and 4.2, \( U \in \text{PO}(X, x) \) and \( \text{pCl}(U) \subseteq \text{pCl}(X_0(U)) \). Hence, we have \( f(\text{pCl}(U)) = (f/X_0)(\text{pCl}(U)) \subseteq (f/X_0)(\text{pCl}(X_0(U))) \subseteq \text{Cl}(V) \). This shows that \( f \) is \( \theta \)-p.c.

**Corollary 4.5.** Let \( \{U_\lambda : \lambda \in \Lambda \} \) be an \( \alpha \)-open cover of a topological space \( X \). A function \( f : X \rightarrow Y \) is \( \theta \)-p.c. if and only if the restriction \( f/U_\lambda : U_\lambda \rightarrow Y \) is \( \theta \)-p.c. for each \( \lambda \in \Lambda \).

**Proof.** This is an immediate consequence of Theorems 4.3 and 4.4.

Let \( \{X_\alpha : \alpha \in \mathcal{A} \} \) be a family of topological spaces, \( A_\alpha \) a nonempty subset of \( X_\alpha \) for each \( \alpha \in \mathcal{A} \) and \( X = \prod \{X_\alpha : \alpha \in \mathcal{A} \} \) denote the product space, where \( \mathcal{A} \) is nonempty.

**Lemma 4.6** (see [8]). Let \( n \) be a positive integer and \( A = \Pi_{j=1}^n A_\alpha_j \times \prod_{\alpha \neq j} X_\alpha \).

1. \( A \in \text{PO}(X) \) if and only if \( A_\alpha_j \in \text{PO}(X_\alpha_j) \) for each \( j = 1, 2, \ldots, n \).
2. \( \text{pCl}(\prod_{\alpha \in \mathcal{A}} A_\alpha) \subseteq \prod_{\alpha \in \mathcal{A}} \text{pCl}(A_\alpha) \).

**Theorem 4.7.** If a function \( f_\alpha : X_\alpha \rightarrow Y_\alpha \) is \( \theta \)-p.c. for each \( \alpha \in \mathcal{A} \). Then the product function \( f : \prod X_\alpha \rightarrow \prod Y_\alpha \), defined by \( f(\{x_\alpha \}) = \{f_\alpha(x_\alpha)\} \) for each \( x = \{x_\alpha \} \), is \( \theta \)-p.c.

**Proof.** Let \( x = \{x_\alpha \} \in \Pi X_\alpha \) and \( W \) be any open set of \( \Pi Y_\alpha \) containing \( f(x) \). Then, there exists an open set \( V_\alpha_j \) of \( Y_\alpha \) such that

\[
f(x) = \{f_\alpha(x_\alpha)\} \subseteq \prod_{j=1}^n V_\alpha_j \times \prod_{\alpha \neq j} Y_\alpha \subset W.
\]

Since \( f_\alpha \) is \( \theta \)-p.c. for each \( \alpha \), there exists \( U_\alpha_j \in \text{PO}(X_\alpha_j, x_\alpha_j) \) such that \( f_\alpha_j(\text{pCl}(U_\alpha_j)) \subset \text{Cl}(V_\alpha_j) \) for \( j = 1, 2, \ldots, n \). Now, put \( U = \prod_{j=1}^n U_\alpha_j \times \prod_{\alpha \neq j} X_\alpha \). Then, it follows from Lemma 4.6 that \( U \in \text{PO}(\Pi X_\alpha, x) \). Moreover, we have

\[
f(\text{pCl}(U)) \subset f(\Pi_{j=1}^n U_\alpha_j) \times \prod_{\alpha \neq j} Y_\alpha
c\subset f(\Pi_{j=1}^n f_\alpha_j(\text{Cl}(U_\alpha_j))) \times \prod_{\alpha \neq j} Y_\alpha
\]

This shows that \( f \) is \( \theta \)-p.c.

**5. Preservation property**

**Definition 5.1.** A topological space \( X \) is said to be

1. \( p \)-closed [7] (resp., \( p \)-Lindelöf) if every cover of \( X \) by preopen sets has a finite (resp., countable) subfamily whose preclosures cover \( X \);
2. countably \( p \)-closed if every countable cover of \( X \) by preopen sets has a finite subfamily whose preclosures cover \( X \);
3. quasi \( H \)-closed [25] (resp., almost Lindelöf [6]) if every cover of \( X \) by open sets has a finite (resp., countable) subfamily whose closures cover \( X \);
4. lightly compact [5] if every countable cover of \( X \) by open sets has a finite subfamily whose closures cover \( X \).
**Definition 5.2.** A subset $K$ of a space $X$ is said to be

1. $p$-closed relative to $X$ [7] if for every cover $\{V_\alpha : \alpha \in \mathcal{A}\}$ of $K$ by preopen sets of $X$, there exists a finite subset $\mathcal{A}_K$ of $\mathcal{A}$ such that $K \subset \bigcup \{pCl(V_\alpha) : \alpha \in \mathcal{A}_K\}$,

2. quasi $H$-closed relative to $X$ [25] if for every cover $\{V_\alpha : \alpha \in \mathcal{A}\}$ of $K$ by open sets of $X$, there exists a finite subset $\mathcal{A}_K$ of $\mathcal{A}$ such that $K \subset \bigcup \{Cl(V_\alpha) : \alpha \in \mathcal{A}_K\}$.

**Theorem 5.3.** If $f : X \to Y$ is a $\theta.p.c.$ function and $K$ is $p$-closed relative to $X$, then $f(K)$ is quasi $H$-closed relative to $Y$.

**Proof.** Suppose that $f : X \to Y$ is $\theta.p.c.$ and $K$ is $p$-closed relative to $X$. Let $\{V_\alpha : \alpha \in \mathcal{A}\}$ be a cover of $f(K)$ by open sets of $Y$. For each point $x \in K$, there exists $\alpha(x) \in \mathcal{A}$ such that $f(x) \in V_{\alpha(x)}$. Since $f$ is $\theta.p.c.$, there exists $U_x \in PO(X,x)$ such that $f(pCl(U_x)) \subset Cl(V_{\alpha(x)})$. The family $\{U_x : x \in K\}$ is a cover of $K$ by preopen sets of $X$ and hence there exists a finite subset $\mathcal{A}_K$ of $K$ such that $K \subset \bigcup_{x \in \mathcal{A}_K} pCl(U_x)$. Therefore, we obtain $f(K) \subset \bigcup_{x \in \mathcal{A}_K} Cl(V_{\alpha(x)})$. This shows that $f(K)$ is quasi $H$-closed relative to $Y$.

**Corollary 5.4.** Let $f : X \to Y$ be a $\theta.p.c.$ surjection. Then, the following properties hold:

1. If $X$ is $p$-closed, then $Y$ is quasi $H$-closed.
2. If $X$ is $p$-Lindelöf, then $Y$ is almost Lindelöf.
3. If $X$ is countably $p$-closed, then $Y$ is lightly compact.

A subset $S$ of a topological space $X$ is said to be $\beta$-open [1] or semipreopen [3] if $S \subset Cl(\text{Int}(Cl(S)))$. It is well known that $\alpha$-openness implies both preopenness and semi-openness which imply $\beta$-openness. The complement of a semipreopen set is said to be $\text{spCl}$-closed [3]. The intersection of all spCl-closed sets of $X$ containing a subset $S$ is the $\text{spCl}$-closure of $S$ and is denoted by $\text{spCl}(S)$ [3].

**Definition 5.5.** A topological space $X$ is said to be

1. $\beta$-connected [24] or semipreconnected [2] if $X$ cannot be expressed as the union of two nonempty disjoint $\beta$-open sets,

2. semi-connected [22] if $X$ cannot be expressed as the union of two nonempty disjoint semi-open sets.

**Remark 5.6.** We have the following implications:

\[ \beta\text{-connected} \Rightarrow \text{semi-connected} \Rightarrow \text{connected}. \]  \hspace{1cm} (5.1)

But, the converses need not be true as the following simple examples show.

**Example 5.7.** (1) Let $X = \{a,b,c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$. Then $(X, \tau)$ is connected but not semi-connected.

(2) Let $X = \{a,b,c\}$ and $\tau = \{X, \emptyset, \{b,c\}\}$. Then $(X, \tau)$ is semi-connected but not $\beta$-connected.

**Lemma 5.8.** For a topological space $X$, the following properties are equivalent:

1. $X$ is $\beta$-connected or semipreconnected.
2. The intersection of two nonempty semipreopen subsets of $X$ is always nonempty.
3. The intersection of two nonempty preopen subsets of $X$ is always nonempty.
(4) \( p\text{Cl}(V) = X \) for every nonempty preopen subset \( V \) of \( X \).
(5) \( sp\text{Cl}(V) = X \) for every nonempty semipreopen subset \( V \) of \( X \).

**Proof.** The proofs of equivalences of (1), (2), and (3) are given in [2, Theorem 6.4]. The other properties (4) and (5), which are stated in [18], are easily equivalent to (3) and (2), respectively. \( \square \)

**Theorem 5.9.** If \( f : X \to Y \) is a \( \theta \)-p.c. surjection and \( X \) is \( \beta \)-connected, then \( Y \) is semi-connected.

**Proof.** Let \( V \) be any nonempty open set of \( Y \). Let \( y \in V \). Since \( f \) is surjective, there exists \( x \in X \) such that \( f(x) = y \). Since \( f \) is \( \theta \)-p.c., there exists \( U \in PO(X,x) \) such that \( f(p\text{Cl}(U)) \subseteq \text{Cl}(V) \). Since \( X \) is \( \beta \)-connected, by Lemma 5.8 \( p\text{Cl}(U) = X \) and hence \( \text{Cl}(V) = Y \) since \( f \) is surjective. Therefore, it follows from [22, Theorem 4.3] that \( Y \) is semi-connected. \( \square \)

**Remark 5.10.** The following example shows that the image of \( \beta \)-connectedness under weakly precontinuous surjections is not necessarily semi-connected.

**Example 5.11.** Let \( X \) be the set of real numbers, \( \tau = \{\emptyset\} \cup \{V \subseteq X : 0 \in V\} \), \( Y = \{a, b, c\} \), and \( \sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\} \). Define a function \( f : (X, \tau) \to (Y, \sigma) \) as follows: \( f(x) = a \) if \( x < 0 \); \( f(x) = c \) if \( x = 0 \); \( f(x) = b \) if \( x > 0 \). Then \( f \) is a weakly precontinuous surjection which is not \( \theta \)-p.c. The topological space \( (X, \tau) \) is \( \beta \)-connected by Lemma 5.8. By Example 5.7(1), \( (Y, \sigma) \) is connected but not semi-connected.

**References**

Zbl 577.54008.


Zbl 604.54002.


Zbl 089.17601.


Zbl 524.54016.


Zbl 138.17601.

292 TAKASHI NOIRI


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Space dynamics is a very general title that can accommodate a long list of activities. This kind of research started with the study of the motion of the stars and the planets back to the origin of astronomy, and nowadays it has a large list of topics. It is possible to make a division in two main categories: astronomy and astrodynamics. By astronomy, we can relate topics that deal with the motion of the planets, natural satellites, comets, and so forth. Many important topics of research nowadays are related to those subjects. By astrodynamics, we mean topics related to spaceflight dynamics.

It means topics where a satellite, a rocket, or any kind of man-made object is travelling in space governed by the gravitational forces of celestial bodies and/or forces generated by propulsion systems that are available in those objects. Many topics are related to orbit determination, propagation, and orbital maneuvers related to those spacecrafts. Several other topics that are related to this subject are numerical methods, nonlinear dynamics, chaos, and control.

The main objective of this Special Issue is to publish topics that are under study in one of those lines. The idea is to get the most recent researches and published them in a very short time, so we can give a step in order to help scientists and engineers that work in this field to be aware of actual research. All the published papers have to be peer reviewed, but in a fast and accurate way so that the topics are not outdated by the large speed that the information flows nowadays.

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<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>July 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>October 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>January 1, 2010</td>
</tr>
</tbody>
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