A weak form of weak quasi-continuity, which we call subweak quasi-continuity, is introduced. It is shown that subweak quasi-continuity is strictly weaker than weak quasi-continuity. Subweak quasi-continuity is used to strengthen several results in the literature concerning weak quasi-continuity. Specifically, results concerning the graph, graph function, and restriction of a weakly quasi-continuous function are extended slightly. Also, a result concerning weakly quasi-continuous retractions is strengthened.

2000 Mathematics Subject Classification: 54C10.

1. Introduction. Weakly quasi-continuous functions were introduced by Popa and Stan [9]. Recently, weak quasi-continuity has been developed further by Noiri [5, 6] and Park and Ha [8]. Due to a result by Noiri [5], weak quasi-continuity is equivalent to the weak semicontinuity developed by Arya and Bhamini [1]. The purpose of this note is to introduce the concept of subweak quasi-continuity, which we define in terms of a base for the topology on the codomain. We establish that this condition is strictly weaker than weak quasi-continuity and we use it to strengthen some of the results in the literature concerning weak quasi-continuity. For example, we show that the graph of a subweakly quasi-continuous function with a Hausdorff codomain is semiclosed. We also show that, if the graph function is subweakly quasi-continuous with respect to the usual base for the product space, then the function itself is weakly quasi-continuous, and that, if a function is subweakly quasi-continuous with respect to the base $\mathcal{B}$, then the restriction to a preopen set is subweakly quasi-continuous with respect to the same base. These results strengthen slightly the comparable results for weakly quasi-continuous functions. Finally, we extend a result concerning weakly quasi-continuous retractions and investigate some of the basic properties of subweakly quasi-continuous functions.

2. Preliminaries. The symbols $X$ and $Y$ denote topological spaces with no separation axioms assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set $A$ are signified by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. A set $A$ is semiopen (preopen, $\alpha$-open) provided that $A \subseteq \text{Cl}(\text{Int}(A))$ ($A \subseteq \text{Int}(\text{Cl}(A))$, $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$). A set is semiclosed (preclosed, $\alpha$-closed) provided that its complement is semiopen (preopen, $\alpha$-open). The collection of all semiopen sets in a space $X$ is denoted by $\text{SO}(X)$ and the collection of all semiclosed sets in $X$ containing a fixed point $x$ is denoted by $\text{SO}(X,x)$. The intersection of all semiclosed sets containing a set $A$ is called the semiclosure of $A$ and denoted by $\text{sCl}(A)$. The
semi-interior of a set $A$, denoted by $\text{sInt}(A)$, is the union of all semiopen sets contained in $A$. The preclosure of $A$, denoted by $\text{pCl}(A)$, is the intersection of all preclosed sets containing $A$. Finally, if an operator is used with respect to a proper subspace, a subscript is added to the operator. Otherwise, it is assumed that the operator refers to the entire space.

**Definition 2.1** (Popa and Stan [9]). A function $f : X \rightarrow Y$ is said to be weakly quasi-continuous if for every $x \in X$, every open set $U$ in $X$ containing $x$, and every open set $V$ in $Y$ containing $f(x)$, there exists a nonempty open set $W$ in $X$ such that $W \subseteq U$ and $f(W) \subseteq \text{Cl}(V)$.

**Definition 2.2** (Arya and Bhamini [1]). A function $f : X \rightarrow Y$ is said to be weakly semicontinuous if for every $x \in X$ and every open set $V$ in $Y$ containing $f(x)$, there exists $U \in \text{SO}(X,x)$ for which $f(U) \subseteq \text{Cl}(V)$.

The following result by Noiri [5] shows that weak quasi-continuity and weak semicontinuity are equivalent.

**Theorem 2.3** (Noiri [5, Theorem 4.1]). A function $f : X \rightarrow Y$ is weakly quasi-continuous if and only if for every $x \in X$ and every open set $V$ containing $f(x)$, there exists $U \in \text{SO}(X,x)$ for which $f(U) \subseteq \text{Cl}(V)$.

**Definition 2.4**. A function $f : X \rightarrow Y$ is said to be subweakly continuous (Rose [10]) (subalmost weakly continuous (Baker [2])) if there is an open base $\mathcal{B}$ for the topology on $Y$ such that $\text{sCl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V))$ for every $V \in \mathcal{B}$.

3. **Subweakly quasi-continuous functions.** The following characterization of weak quasi-continuity is due to Noiri [5].

**Theorem 3.1** (Noiri [5, Theorem 4.3(d)]). A function $f : X \rightarrow Y$ is weakly quasi-continuous if and only if $\text{sCl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V))$ for every open set $V$ in $Y$.

We define a function $f : X \rightarrow Y$ to be subweakly quasi-continuous provided that there is an open base $\mathcal{B}$ for the topology on $Y$ for which $\text{sCl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V))$ for every $V \in \mathcal{B}$. Obviously, weak quasi-continuity implies subweak quasi-continuity. The following example shows that these concepts are not equivalent.

**Example 3.2.** Let $X = \mathbb{R}$ have the usual topology and $Y = X$ have the discrete topology. The identity mapping $f : X \rightarrow Y$ is subweakly quasi-continuous with respect to the base consisting of the singleton sets in $Y$. However, $f$ is not weakly quasi-continuous because for $V = (0,1) \cup (1,2)$, $\text{sCl}(f^{-1}(V)) \not\subseteq f^{-1}(\text{Cl}(V))$.

Since $\text{sCl}(A) = A \cup \text{Int}(\text{Cl}(A))$ for every set $A$, we have the following characterization of subweak quasi-continuity.

**Theorem 3.3.** A function $f : X \rightarrow Y$ is subweakly quasi-continuous if and only if there is an open base $\mathcal{B}$ for the topology on $Y$ for which $\text{Int}(\text{Cl}(f^{-1}(V))) \subseteq f^{-1}(\text{Cl}(V))$ for every $V \in \mathcal{B}$.
Since $s\text{Cl}(A) \subseteq \text{Cl}(A)$ for every set $A$, obviously, subweak continuity implies subweak quasi-continuity. The following example shows that the converse implication does not hold.

**Example 3.4.** Let $X = [1/2, 3/2]$ have the usual relative topology, $Y = \{0, 1\}$ have the discrete topology, and let $f : X \to Y$ be the greatest integer function. Kar and Battacharya [3] showed that $f$ is weakly quasi-continuous (their term is weakly semicontinuous) but not weakly continuous. Obviously, the function $f$ is also not subweakly continuous.

The following two examples establish that subweak quasi-continuity is independent of subalmost weak continuity.

**Example 3.5.** Let $X$ be an indiscrete space with at least two elements and let $Y = X$ have the discrete topology. Since $p\text{Cl}(\{x\}) = \{x\}$ for every $x \in X$, the identity mapping $f : X \to Y$ is subalmost weakly continuous with respect to the base consisting of the singleton sets in $Y$. However, since singleton sets in $X$ are dense, $f$ is not subweakly quasi-continuous.

**Example 3.6.** Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $Y = X$ have the discrete topology. Let $f : X \to Y$ be the identity mapping. The function $f$ is not subalmost weakly continuous, since any base for $Y$ must include $V = \{a\}$ and $p\text{Cl}(f^{-1}(V)) \notin f^{-1}(\text{Cl}(V))$. However, $f$ is subweakly quasi-continuous with respect to the base of singleton subsets of $Y$.

**4. Graph related properties.** Recall that the graph of a function $f : X \to Y$ is the subspace $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $X \times Y$.

Park and Ha [8] proved that the graph of a weakly quasi-continuous function with a Hausdorff codomain is semiclosed. We show that weak quasi-continuity can be replaced by subweak quasi-continuity.

**Theorem 4.1.** If $f : X \to Y$ is subweakly quasi-continuous and $Y$ is Hausdorff, then the graph of $f$, $G(f)$, is semiclosed.

**Proof.** Let $\mathcal{B}$ be an open base for $Y$ such that $s\text{Cl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V))$ for every $V \in \mathcal{B}$. Let $(x, y) \in X \times Y - G(f)$. Since $y \neq f(x)$, there exists disjoint open sets $V$ and $W$ with $f(x) \in W$, $y \in V$, and $V \in \mathcal{B}$. Then $x \notin f^{-1}(\text{Cl}(V))$, and, since $s\text{Cl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V))$, $x \notin s\text{Cl}(f^{-1}(V))$. Therefore $(x, y) \in (X - s\text{Cl}(f^{-1}(V))) \times V \subseteq X \times Y - G(f)$. Since $s\text{Cl}(f^{-1}(V))$ is semiclosed, $X - s\text{Cl}(f^{-1}(V))$ is semiopen. Since finite products of semiopen sets are semiopen, $(X - s\text{Cl}(f^{-1}(V))) \times V$ is semiopen. Finally, since unions of semiopen sets are semiopen, it follows that $X \times Y - G(f)$ is semiopen and that $G(f)$ is semiclosed.

**Corollary 4.2** (Park and Ha [8, Corollary 4.2]). If $f : X \to Y$ is weakly quasi-continuous and $Y$ is Hausdorff, then the graph of $f$, $G(f)$, is semiclosed.

By the graph function of a function $f : X \to Y$ we mean the function $g : X \to X \times Y$ given by $g(x) = (x, f(x))$ for every $x \in X$. 

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Theorem 4.3. Let \( f : (X, \tau) \to (Y, \sigma) \) be a function and let \( \mathcal{B} \) be an open base for \( \sigma \). Let \( \mathcal{C} = \{ U \times V : U \in \tau, V \in \mathcal{B} \} \). The function \( f \) is subweakly quasi-continuous with respect to the base \( \mathcal{B} \) if and only if the graph function of \( f, g : X \to X \times Y \), is subweakly quasi-continuous with respect to the base \( \mathcal{C} \).

Proof. Assume that \( f : (X, \tau) \to (Y, \sigma) \) is subweakly quasi-continuous with respect to the base \( \mathcal{B} \) for \( \sigma \). Let \( U \times V \in \mathcal{C} \), where \( U \in \tau \) and \( V \in \mathcal{B} \). Then
\[
\text{scCl}(f^{-1}(U \times V)) = \text{scCl}(U \cap f^{-1}(V)) \subseteq \text{Cl}(U) \cap \text{scCl}(f^{-1}(V)) \subseteq \text{Cl}(U) \cap f^{-1}(\text{Cl}(V)) = g^{-1}(\text{Cl}(U) \times \text{Cl}(V)) = g^{-1}(\text{Cl}(U \times V)).
\]
Therefore, \( f \) is subweakly quasi-continuous with respect to the base \( \mathcal{B} \).

In Theorem 4.3, if we take \( \mathcal{B} \) to be \( \sigma \), the topology on \( Y \), then we have the following result.

Corollary 4.4. If the graph function \( g : X \to X \times Y \) of a function \( f \) is subweakly quasi-continuous with respect to the usual base for the product space, then the function \( f \) is weakly quasi-continuous.

Corollary 4.5 (Noiri [5, The “only if” part of Theorem 6.3.4]). If the graph function \( g : X \to X \times Y \) of a function \( f \) is weakly quasi-continuous, then the function \( f \) is weakly quasi-continuous.

5. Additional properties

Definition 5.1 (Kar and Bhattacharya [4]). A space \( X \) is said to be semi-\( T_1 \) provided that for every pair of distinct points \( x \) and \( y \) in \( X \) there exist sets \( U \in \text{SO}(X, x) \) and \( V \in \text{SO}(X, y) \) such that \( y \notin U \) and \( x \notin V \).

Theorem 5.2. If \( Y \) is Hausdorff and \( f : X \to Y \) is a subweakly quasi-continuous injection, then \( X \) is semi-\( T_1 \).

Proof. Let \( x_1 \) and \( x_2 \) be distinct points in \( X \) and let \( \mathcal{B} \) be an open base for \( Y \) such that \( \text{scCl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V)) \) for every \( V \in \mathcal{B} \). Since \( Y \) is Hausdorff and \( f(x_1) \neq f(x_2) \), there exist disjoint open sets \( U \) and \( V \) in \( Y \) with \( f(x_1) \in U \) and \( f(x_2) \in V \), and \( V \in \mathcal{B} \). Then, since \( f(x_1) \notin \text{Cl}(V) \), we have \( x_1 \in X - f^{-1}(\text{Cl}(V)) \subseteq X - \text{scCl}(f^{-1}(V)) \) which is semiopen and does not contain \( x_2 \). Therefore \( X \) is semi-\( T_1 \).

The function in Example 3.6 is a subweakly quasi-continuous injection with a Hausdorff codomain and a non-\( T_1 \)-domain. Therefore, the conclusion that \( X \) is semi-\( T_1 \) in Theorem 5.2 cannot be strengthened to \( T_1 \).

Since the restriction of the function \( f \) in Example 3.6 to the set \( A = \{ a, c \} \) is not subweakly quasi-continuous, we see that the restriction of a subweakly quasi-continuous function can fail to be subweakly quasi-continuous. Noiri [5] proved that the restriction of weakly quasi-continuous function to an open set is weakly quasi-continuous and Arya and Bhamini [1] extended this result to \( \alpha \)-open sets. Finally, Park and Ha [8]
extended the result further to preopen sets. In what follows, we establish the analogous result for subweakly quasi-continuous functions.

**Theorem 5.3.** If \( f : X \to Y \) is subweakly quasi-continuous with respect to the base \( \mathcal{B} \) for \( Y \) and \( A \) is a preopen set in \( X \), then \( f|_A : A \to Y \) is subweakly quasi-continuous with respect to the base \( \mathcal{B} \).

**Proof.** Let \( V \in \mathcal{B} \), then using (Noiri [7, Lemma 3.3]) we see that \( s\text{Cl}_A(f|_A^{-1}(V)) = A \cap s\text{Cl}(f|_A^{-1}(V)) = A \cap \text{Cl}(f^{-1}(V) \cap A) \subseteq A \cap \text{Cl}(f^{-1}(V)) \subseteq A \cap f^{-1}(\text{Cl}(V)) = f|_A^{-1}(\text{Cl}(V)) \). Therefore, \( f|_A : A \to Y \) is subweakly quasi-continuous with respect to the base \( \mathcal{B} \).

In Theorem 5.3, if we let \( \mathcal{B} \) be the topology, then we have the following result.

**Corollary 5.4** (Park and Ha [8, Theorem 3.8]). If \( f : X \to Y \) is weakly quasi-continuous and \( A \) is a preopen set in \( X \), then \( f|_A : A \to Y \) is weakly quasi-continuous.

**Theorem 5.5.** If \( f : X \to Y \) is subweakly quasi-continuous and \( A \) is an open set in \( Y \) containing \( f(X) \), then \( f : X \to A \) is subweakly quasi-continuous.

**Proof.** Let \( \mathcal{B} \) be an open base for \( Y \) for which \( s\text{Cl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V)) \) for every \( V \in \mathcal{B} \). Then \( \mathcal{C} = \{ V \cap A : V \in \mathcal{B} \} \) is an open base for the relative topology on \( A \). Let \( V \cap A \in \mathcal{C} \), where \( V \in \mathcal{B} \). Then \( s\text{Cl}(f^{-1}(V \cap A)) = s\text{Cl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V)) \subseteq f^{-1}(\text{Cl}(V) \cap A) \). The proof is completed by establishing that \( \text{Cl}(V) \cap A \subseteq \text{Cl}_A(V \cap A) \).

Let \( y \in \text{Cl}(V) \cap A \) and let \( W \subseteq A \) be open in \( A \) with \( y \in W \). Since \( A \) is open in \( Y \), \( W \) is open in \( Y \). Because \( y \in \text{Cl}(V) \), \( W \cap V \neq \emptyset \). Therefore \( W \cap (V \cap A) \neq \emptyset \), which proves that \( y \in \text{Cl}_A(V \cap A) \). Thus \( \text{Cl}(V) \cap A \subseteq \text{Cl}_A(V \cap A) \).

Now, it follows that \( f : X \to A \) is subweakly quasi-continuous.

Park and Ha [8] defined a function \( f : X \to A \), where \( A \subseteq X \), to be a weakly quasi-continuous retraction provided that \( f \) is weakly quasi-continuous and \( f|_A \) is the identity on \( A \). It is then proved (Park and Ha [8, Theorem 3.15]) that, if \( f : X \to A \) is a weakly quasi-continuous retraction and \( X \) is Hausdorff, then \( A \) is semiclosed in \( X \). We prove the following comparable result for subweakly quasi-continuous functions.

**Theorem 5.6.** Let \( A \subseteq X \) and let \( f : X \to X \) be a subweakly quasi-continuous function such that \( f(X) = A \) and \( f|_A \) is the identity on \( A \). If \( X \) is Hausdorff, then \( A \) is semiclosed.

**Proof.** Assume \( A \) is not semiclosed. Let \( x \in \text{Cl}(A) - A \). Let \( \mathcal{B} \) be an open base for the topology on \( X \) such that \( s\text{Cl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V)) \) for every \( V \in \mathcal{B} \). Since \( x \notin A \), \( f(x) \notin f(x) \). Because \( X \) is Hausdorff, there exist disjoint open sets \( V \) and \( W \) such that \( x \in V \), \( f(x) \in W \), and \( V \in \mathcal{B} \). Let \( U \in \text{SO}(X,x) \). Then \( x \in U \cap V \), which is semiopen in \( X \) (Noiri [7]). Since \( x \in \text{Cl}(A) \), \( (U \cap V) \cap A \neq \emptyset \). So there exists \( y \in (U \cap V) \cap A \). Since \( y \in A \), \( f(y) = y \) and therefore \( y \in f^{-1}(V) \). Thus \( U \cap f^{-1}(V) \neq \emptyset \) and we see that \( x \in s\text{Cl}(f^{-1}(V)) \). However, \( f(x) \in W \), which is open and disjoint from \( V \). Hence \( f(x) \notin \text{Cl}(V) \) or \( x \notin f^{-1}(\text{Cl}(V)) \), which contradicts the assumption that \( f \) is subweakly quasi-continuous.

**Lemma 5.7.** If \( A \subseteq Y \) and \( f : X \to A \) is weakly quasi-continuous, then \( f : X \to Y \) is weakly quasi-continuous.
**Proof.** If $V$ is an open set in $Y$, then $\text{sCl}(f^{-1}(V)) = s\text{Cl}(f^{-1}(V \cap A)) \subseteq f^{-1}((\text{Cl}_A(V \cap A)) = f^{-1}(A \cap \text{Cl}(V \cap A)) = f^{-1}(\text{Cl}(V \cap A)) \subseteq f^{-1}(\text{Cl}(V))$. \hfill $\Box$

Thus a weakly quasi-continuous retraction satisfies the hypothesis of Theorem 5.6 and we have the following corollary.

**Corollary 5.8** (Park and Ha [8, Theorem 3.15]). If $f : X \to A$, where $A \subseteq X$, is a weakly quasi-continuous retraction and $X$ is Hausdorff, then $A$ is semiclosed.

**Theorem 5.9.** Let $Y$ be a Hausdorff space, $f_1 : X \to Y$ continuous, and $f_2 : X \to Y$ subweakly quasi-continuous. Then $\{x \in X : f_1(x) = f_2(x)\}$ is semiclosed.

**Proof.** Let $A = \{x \in X : f_1(x) = f_2(x)\}$ and let $x \in X - A$. Let $\mathcal{B}$ be an open base for the topology on $Y$ for which $s\text{Cl}(f_2^{-1}(V)) \subseteq f_2^{-1}(\text{Cl}(V))$ for every $V \in \mathcal{B}$. Since $Y$ is Hausdorff and $f_1(x) \neq f_2(x)$, there exist disjoint open sets $V$ and $W$ in $Y$ for which $f_1(x) \in V$, $f_2(x) \in W$, and $V \cap W = \emptyset$. Since $f_2(x) \notin \text{Cl}(V)$, we have $x \in X - f_2^{-1}(\text{Cl}(V)) \subseteq X - s\text{Cl}(f_2^{-1}(V))$. Therefore $x \in f_1^{-1}(V) \cap (X - s\text{Cl}(f_2^{-1}(V))) \subseteq X - A$. Since $f_1^{-1}(V)$ is open, $X - s\text{Cl}(f_2^{-1}(V))$ is semiopen, and the intersection of an open set and a semiopen set is semiopen (Noiri [7]), we see that $X - A$ is semiopen and that $A$ is semiclosed. \hfill $\Box$

**Corollary 5.10.** Let $Y$ be Hausdorff, $f_1 : X \to Y$ continuous, and $f_2 : X \to Y$ subweakly quasi-continuous. If $f_1$ and $f_2$ agree on a dense subset of $X$, then $f_1 = f_2$.

**Acknowledgment.** The author gratefully acknowledges the support of Indiana University Southeast in the publication of this paper.

**References**


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