ON A CLASS OF MEROMORPHIC $p$-VALENT STARLIKE FUNCTIONS INVOLVING CERTAIN LINEAR OPERATORS

JIN-LIN LIU and SHIGEYOSHI OWA

Received 10 March 2002

Let $\Sigma_p$ be the class of functions $f(z)$ which are analytic in the punctured disk $E^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Applying the linear operator $D^{n+p}$ defined by using the convolutions, the subclass $\mathcal{F}_n^{p}(\alpha)$ of $\Sigma_p$ is considered. The object of the present paper is to prove that $\mathcal{F}_n^{p}(\alpha) \subset \mathcal{F}_n^{p-1}(\alpha)$, Since $\mathcal{F}_0^{p}(\alpha)$ is the class of meromorphic $p$-valent starlike functions of order $\alpha$, all functions in $\mathcal{F}_n^{p-1}(\alpha)$ are meromorphic $p$-valent starlike in the open unit disk $E$. Further properties preserving integrals and convolution conditions are also considered.

2000 Mathematics Subject Classification: 30C45.

1. Introduction. Let $\Sigma_p$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{m=1-p}^{\infty} a_m z^m \quad (p \in \mathbb{N} = \{1, 2, \ldots\}), \quad (1.1)$$

which are analytic in the punctured disk $E^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. The convolution of two power series $f(z)$, given by (1.1) and

$$g(z) = \frac{1}{z^p} + \sum_{m=1-p}^{\infty} b_m z^m, \quad (1.2)$$

is defined as the following power series:

$$(f * g)(z) = \frac{1}{z^p} + \sum_{m=1-p}^{\infty} a_m b_m z^m. \quad (1.3)$$

Let $\mathcal{F}_p^{*}(\alpha)$ denote the class of functions of the form (1.1), which satisfy the condition

$$\text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} < -p\alpha \quad (z \in E = \{z \in \mathbb{C} : |z| < 1\}) \quad (1.4)$$

for some $\alpha$ ($0 \leq \alpha < 1$). A function $f(z)$ in $\mathcal{F}_p^{*}(\alpha)$ is called a meromorphic $p$-valent starlike of order $\alpha$ in $E$.

A function $f(z) \in \Sigma_p$ is said to be in the class $\mathcal{F}_n^{p-1}(\alpha)$ if it satisfies the inequality

$$\text{Re}\left\{ \frac{D^{n+p}f(z) - n + 2p}{D^{n+p-1}f(z)} - \frac{p\alpha}{n+p} \right\} < -\frac{p\alpha}{n+p} \quad (z \in E), \quad (1.5)$$
where \( n \) is any integer greater than \(-p\), \( 0 \leq \alpha < 1 \), and

\[
D^{n+p-1} f(z) = \frac{1}{z^p(1-z)^{n+p}} f(z) = \frac{(z^{n+2p-1} f(z))^{(n+p-1)}}{(n+p-1)! z^p} = \frac{1}{z^p} \sum_{m=1-p}^{\infty} \frac{(m+n+2p-1)!}{(n+p-1)!(m+p)} a_m z^m. \tag{1.6}
\]

The operator \( D^{n+p-1} \) when \( p = 1 \) was first introduced by Ganigi and Uralegaddi [1] and then generalized by Yang [9]. In recent years, many authors (e.g., [8, 10, 11]) have investigated certain subclasses of meromorphic functions defined by the operator \( D^{n+p-1} \). In this paper, we show that a function \( f(z) \) belonging to \( \mathcal{T}_{n+p-1}(\alpha) \) is meromorphic \( p \)-valent starlike of order \( \alpha \). More precisely, it is proved that

\[
\mathcal{T}_{n+p}(\alpha) \subset \mathcal{T}_{n+p-1}(\alpha). \tag{1.7}
\]

Since \( \mathcal{T}_0(\alpha) \) is the class of functions that satisfy the condition

\[
\text{Re} \frac{zf'(z)}{f(z)} < -p\alpha \quad (z \in \mathbb{E}), \tag{1.8}
\]

the starlikeness of members of \( \mathcal{T}_{n+p-1}(\alpha) \) is a consequence of (1.7). Further, integral transforms of functions in \( \mathcal{T}_{n+p-1}(\alpha) \) and convolution conditions are also considered.

2. Properties of the class \( \mathcal{T}_{n+p-1}(\alpha) \). In proving our main results, we will need the following lemma.

**Lemma 2.1** (see [2, 3]). Let \( w(z) \) be nonconstant and analytic in \( \mathbb{E} \), \( w(0) = 0 \). If \( |w(z)| \) attains its maximum value on the circle \( |z| = r < 1 \) at \( z_0 \), then \( z_0 w'(z_0) = kw(z_0) \), where \( k \) is a real number and \( k \geq 1 \).

**Theorem 2.2.** We have \( \mathcal{T}_{n+p}(\alpha) \subset \mathcal{T}_{n+p-1}(\alpha) \).

**Proof.** Let \( f(z) \in \mathcal{T}_{n+p}(\alpha) \), then

\[
\text{Re} \left\{ \frac{D^{n+p+1} f(z)}{D^{n+p} f(z)} - \frac{n+2p+1}{n+p+1} \right\} < -\frac{p\alpha}{n+p+1} \quad (z \in \mathbb{E}). \tag{2.1}
\]

We have to show that (2.1) implies the inequality

\[
\text{Re} \left\{ \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} - \frac{n+2p}{n+p} \right\} < -\frac{p\alpha}{n+p} \quad (z \in \mathbb{E}). \tag{2.2}
\]

Consider the analytic function \( w(z) \) in \( \mathbb{E} \) defined by

\[
\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} - \frac{n+2p}{n+p} = -\frac{p}{n+p} \cdot \frac{1 + (1-2\alpha)w(z)}{1-w(z)}. \tag{2.3}
\]
ON A CLASS OF MEROMORPHIC \( p \)-VALENT STARLIKE FUNCTIONS ... 273

It is clear that \( w(0) = 0 \). Equation (2.3) may be written as

\[
\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} = \frac{1 - (1 + 2p(1 - \alpha)/(n + p)) w(z)}{1 - w(z)}.
\]  

(2.4)

Differentiating (2.4) logarithmically and using the identity (easy to verify)

\[
z(D^{n+p-1} f(z))' = (n + p)D^{n+p} f(z) - (n + 2p)D^{n+p-1} f(z),
\]  

(2.5)

we obtain

\[
D^{n+p+1} f(z) - \frac{n + 2p + 1 - p \alpha}{n + p + 1} f(z)
\]

\[
= \frac{1}{n + p + 1} \left\{ - \frac{p(1 - \alpha)(1 + w(z))}{1 - w(z)}
\right.
\]

\[
- \frac{2p(1 - \alpha)}{n + p} \cdot \frac{z w'(z)}{(1 - w(z))(1 - (1 + 2p(1 - \alpha)/(n + p)) w(z))} \left. \right\}
\]  

(2.6)

We claim that \(|w(z)| < 1\) in \( E \). For otherwise, by Lemma 2.1, there exists a point \( z_0 \) in \( E \) such that \( z_0 w'(z_0) = k w(z_0) \), where \(|w(z_0)| = 1\) and \( k \geq 1 \). Equation (2.6) yields

\[
\frac{1}{n + p + 1} \left\{ - \frac{p(1 - \alpha)(1 + w(z_0))}{1 - w(z_0)}
\right.
\]

\[
- \frac{2p(1 - \alpha)}{n + p} \cdot \frac{k w(z_0)}{(1 - w(z_0))(1 - (1 + 2p(1 - \alpha)/(n + p)) w(z_0))} \left. \right\}
\]

\[
\geq \frac{p(1 - \alpha)}{2(n + p + 1)(n + 2p - p \alpha)} > 0,
\]  

(2.7)

which contradicts (2.1). Hence, \(|w(z)| < 1\) in \( E \) and it follows that \( f(z) \in \mathbb{T}_{n+p-1}(\alpha) \).

\[ \square \]

**Theorem 2.3.** Let \( c > 0 \) and let \( f(z) \in \sum_p \) satisfy the condition

\[
\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} - \frac{n + 2p}{n + p} < - \frac{p\alpha}{n + p} + \frac{p(1 - \alpha)}{2(n + p)(c + p - p \alpha)} \quad (z \in E).
\]  

(2.8)

Then,

\[
F(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t) dt
\]  

(2.9)

belongs to \( \mathbb{T}_{n+p-1}(\alpha) \).
**Proof.** From the definition of $F(z)$, we have

$$z(D^{n+p-1}F(z))' = cD^{n+p-1}f(z) - (c + p)D^{n+p-1}F(z).$$  \hfill (2.10)

Using (2.5) and (2.10), condition (2.8) may be written as

$$\Re \left\{ \frac{(n + p + 1)(D^{n+p+1}F(z)/D^{n+p}F(z)) - (n + p + 1 - c)}{(n + p) - (n + p - c)(D^{n+p-1}F(z)/D^{n+p}F(z))} \frac{n + 2p}{n + p} \right\}$$

$$< - \frac{p\alpha}{n + p} + \frac{p(1 - \alpha)}{2(n + p)(c + p - p\alpha)}.$$  \hfill (2.11)

We have to prove that (2.11) implies the inequality

$$\Re \left\{ \frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} - \frac{n + 2p}{n + p} \right\} < - \frac{p\alpha}{n + p}.$$  \hfill (2.12)

Define $w(z)$ in $E$ by

$$\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} - \frac{n + 2p}{n + p} = - \frac{p}{n + p} \cdot \frac{1 + (1 - 2\alpha)w(z)}{1 - w(z)}.$$  \hfill (2.13)

Clearly, $w(z)$ is analytic and $w(0) = 0$. Equation (2.13) may be written as

$$\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} = \frac{1 - (1 + 2p(1 - \alpha)/(n + p))w(z)}{1 - w(z)}.$$  \hfill (2.14)

Differentiating (2.14) logarithmically and using (2.5), we obtain

$$(n + p + 1)\frac{D^{n+p+1}F(z)}{D^{n+p+1}F(z)} - (n + p)\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} - 1$$

$$= - \frac{2p(1 - \alpha)}{n + p} \cdot \frac{zw'(z)}{(1 - w(z))(1 - (1 + 2p(1 - \alpha)/(n + p))w(z))}.$$  \hfill (2.15)

Using (2.14) and (2.15), we get

$$\frac{(n + p + 1)(D^{n+p+1}F(z)/D^{n+p}F(z)) - (n + p + 1 - c)}{(n + p) - (n + p - c)(D^{n+p-1}F(z)/D^{n+p}F(z))} - \frac{n + 2p - p\alpha}{n + p}$$

$$= \frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} - \frac{n + 2p - p\alpha}{n + p}$$

$$- \frac{2p(1 - \alpha)}{n + p} \cdot \frac{zw'(z)}{(1 - w(z))(1 - (n + 3p - 2p\alpha)w(z))}$$

$$= - \frac{p(1 - \alpha)}{n + p} \cdot \frac{1 + w(z)}{1 - w(z)} - \frac{2p(1 - \alpha)}{n + p} \cdot \frac{zw'(z)}{(1 - w(z))(c - (c + 2p - 2p\alpha)w(z))}.$$  \hfill (2.16)

The remaining part of the proof is similar to that of Theorem 2.2. \qed
According to Theorem 2.2, we have the following corollary at once.

**Corollary 2.4.** If \( f(z) \in \mathbb{T}_{n+p-1}(\alpha) \), then the function \( F(z) \) defined by (2.9) also belongs to \( \mathbb{T}_{n+p-1}(\alpha) \).

**Theorem 2.5.** We have \( f(z) \in \mathbb{T}_{n+p-1}(\alpha) \) if and only if

\[
F(z) = \frac{n+p}{2n+2p} \int_0^z t^{n+2p-1} f(t) \, dt \in \mathbb{T}_{n+p}(\alpha).
\]

**Proof.** From the definition of \( F(z) \) we have

\[
z(D^{n+p-1}F(z))' = (n+p)D^{n+p-1}f(z) - (n+2p)D^{n+p-1}F(z).
\]

Using identity (2.5), (2.18) reduces to \( D^{n+p-1}f(z) = D^{n+p}F(z) \). Hence, \( D^{n+p}f(z) = D^{n+p+1}F(z) \) and the result follows.

**Theorem 2.6.** Let \( c > 0 \) and \( 0 \leq \alpha \leq \beta < 1 \). If \( F(z) \in \mathbb{T}_{n+p-1}(\alpha) \), then the function \( f(z) \), defined by (2.9), belongs to \( \mathbb{T}_{n+p-1}(\beta) \) in \( |z| < \rho \), where \( \rho = \min(\rho_1, \rho_2) \), \( \rho_1 = c/(c+2p-2p\alpha) \in (0,1) \), and \( \rho_2 \in (0,1) \), is a minimum positive root of the equation

\[
(1+\beta-2\alpha)(c+2p-2p\alpha)r^3 - [(1-\alpha)(3c+4p-4p\alpha-2) + c(\beta-\alpha)]r^2 \\
+ [2(1-\alpha)(c+1) + (1-\beta)(c+2p-2p\alpha)]r - c(1-\beta) = 0.
\]

**Proof.** If \( F(z) \in \mathbb{T}_{n+p-1}(\alpha) \), then

\[
\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} - \frac{n+2p}{n+p} = -\frac{p}{n+p} [\alpha + (1-\alpha)u(z)],
\]

where \( u(z) \) is analytic in \( \mathbb{E} \) with \( u(0) = 1 \) and \( \text{Re}(u(z)) > 0 \) for \( z \in \mathbb{E} \). Using (2.5), (2.10), and (2.20), we have

\[
(n+p)D^{n+p}f(z) - (n+2p) + p\beta \\
= p(\beta - \alpha) - p(1-\alpha)u(z) + \frac{p(1-\alpha)zu'(z)}{p(1-\alpha)u(z) - (p-p\alpha+c)}.
\]

It is well known that for \( |z| = r < 1 \),

\[
\frac{1-r}{1+r} \leq \text{Re}(u(z)) \leq \frac{1+r}{1-r}, \\
|zu'(z)| \leq \frac{2r}{1-r^2} \text{Re}(u(z)).
\]
Thus, from (2.21) we have for \(|z| = r < \rho_1 = c/(c + 2p - 2p\alpha),\)

\[
\text{Re}\left\{ (n + p) \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} - (n+2p) + p\beta \right\} \leq p(\beta - \alpha) - p(1-\alpha) \text{Re} u(z) + \left| \frac{p(1-\alpha)z u'(z)}{(p-p\alpha+c) - p(1-\alpha)u(z)} \right| \quad (2.23)
\]

\[
\leq \frac{Q(r)}{(1-r^2)[c-(2p-2p\alpha+c)r]},
\]

where

\[
Q(r) = p(\beta - \alpha)(1-r^2)[c-(2p-2p\alpha+c)r] - p(1-\alpha)(1-r)^2[c-(2p-2p\alpha+c)r] + 2p(1-\alpha)(r+r^2).
\]

Since \(Q(r)\) is continuous on \([0,1]\) with \(Q(0) = -cp(1-\beta) < 0\) and \(Q(1) = 4p(1-\alpha) > 0\), (2.19) has a minimum positive root \(\rho_2 \in (0,1)\). This proves that \(f(z)\) belongs to \(\mathcal{F}_{n+p-1}(\beta)\) in \(|z| < \rho\), where \(\rho = \min(\rho_1, \rho_2)\).

To prove Theorem 2.9, we need the following lemmas.

**Lemma 2.7** (see [5]). The function \((1-z)^\beta = e^{\beta \log(1-z)}, \beta \neq 0\), is univalent in \(\mathbb{E}\) if and only if \(\beta\) is either in the closed disk \(|\beta - 1| \leq 1\) or in the closed disk \(|\beta + 1| \leq 1\).

**Lemma 2.8** (see [4]). Let \(q(z)\) be univalent in \(\mathbb{E}\) and let \(Q(w)\) and \(\Phi(w)\) be analytic in a domain \(D\) containing \(q(\mathbb{E})\), with \(\Phi(w) \neq 0\) when \(w \in q(\mathbb{E})\). Set \(Q(z) = zq'(z)\Phi(q(z))\), \(h(z) = \theta(q(z)) + Q(z)\) and suppose that

1. \(Q(z)\) is starlike (univalent) in \(\mathbb{E}\),
2. \(Q(z)\) and \(h(z)\) satisfy

\[
\text{Re} \frac{zh'(z)}{Q(z)} = \text{Re} \left\{ \frac{Q'(q(z)) + zQ'(z)}{\Phi(q(z)) + zQ(z)} \right\} > 0, \quad z \in \mathbb{E}.
\]

If \(p(z)\) is analytic in \(\mathbb{E}\), with \(p(0) = q(0), p(\mathbb{E}) \subset D\), and

\[
\theta(p(z)) + zp'(z)\Phi(p(z)) < \theta(q(z)) + zq'(z)\Phi(q(z)) = h(z),
\]

then \(p(z) < q(z)\), and \(q(z)\) is the best dominant of (2.26).

**Theorem 2.9.** Let \(f(z) \in \mathcal{F}_{n+p-1}(\alpha)\) and let \(\beta\) be a complex number with \(\beta \neq 0\) and satisfy either \(|2p\beta(1-\alpha) - 1| \leq 1\) or \(|2p\beta(1-\alpha) + 1| \leq 1\). Then

\[
(z^pD^{n+p-1} f(z))^\beta < (1-z)2p\beta(1-\alpha) = q(z), \quad z \in \mathbb{E},
\]

and \(q(z)\) is the best dominant.
**Proof.** Set

\[ p(z) = (z^n D^{n+p-1} f(z))^\beta, \quad z \in \mathbb{E}, \tag{2.28} \]

then \( p(z) \) is analytic in \( \mathbb{E} \) with \( p(0) = 1 \). Differentiating \( (2.28) \) logarithmically, we have

\[ \frac{zp'(z)}{p(z)} = \beta \left[ \frac{z(D^{n+p-1} f(z))'}{D^{n+p-1} f(z)} + p \right]. \tag{2.29} \]

Since \( f \in \mathcal{T}_{n+p-1}(\alpha) \), \( (2.29) \) is equivalent to

\[ -p + \frac{zp'(z)}{p(z)} \prec -p \frac{1 + (1-2\alpha)z}{1-z}. \tag{2.30} \]

On the other hand, if we take \( q(z) = (1-z)^{2p\beta(1-\alpha)}, \theta(w) = -p, \) and \( \Phi(w) = 1/\beta w \) in Lemma 2.8, then \( q(z) \) is univalent by the condition of the theorem and Lemma 2.7. It is easy to see that \( q(z), \theta(w), \) and \( \Phi(w) \) satisfy the conditions of Lemma 2.8. Since

\[ Q(z) = zq'(z)\Phi(q(z)) = -\frac{2p(1-\alpha)z}{1-z} \tag{2.31} \]

is univalent starlike in \( \mathbb{E} \) and

\[ h(z) = \theta(q(z)) + Q(z) = -p \frac{1 + (1-2\alpha)z}{1-z}, \tag{2.32} \]

it may be readily checked that conditions (1) and (2) of Lemma 2.8 are satisfied. Thus, the result follows from \( (2.30) \) and Lemma 2.8. \( \square \)

**Corollary 2.10.** Let \( f \in \mathcal{T}_{n+p-1}(\alpha) \). Then

\[ \Re \left[ z^p D^{n+p-1} f(z) \right]^\beta > 2^{2p\beta(1-\alpha)}, \quad z \in \mathbb{E}, \tag{2.33} \]

where \( \beta \) is a real number and \( \beta \in [-1/2p(1-\alpha), 0) \). The result is sharp.

**Proof.** From Theorem 2.9, we have

\[ \Re \left[ z^p D^{n+p-1} f(z) \right]^\beta = \Re \left[ (1-w(z))^{2p\beta(1-\alpha)} \right], \quad z \in \mathbb{E}, \tag{2.34} \]

where \( w(z) \) is analytic in \( \mathbb{E} \), \( w(0) = 0 \), and \( |w(z)| < 1 \) for \( z \in \mathbb{E} \).

In view of \( \Re(t^b) \geq (\Re t)^b \) for \( \Re t > 0 \) and \( 0 < b \leq 1 \), \( (2.34) \) yields

\[ \Re \left[ z^p D^{n+p-1} f(z) \right]^\beta \geq \left[ \Re \frac{1}{1-w(z)} \right]^{-2p\beta(1-\alpha)} > 2^{2p\beta(1-\alpha)}, \quad z \in \mathbb{E}, \tag{2.35} \]

for \(-1 \leq 2p\beta(1-\alpha) < 0 \).
To see that the bound $2^2 p (1-\alpha)$ cannot be increased, we consider the function $f(z)$ which satisfies $z^p D^{n+p-1} f(z) = (1-z)^{2p(1-\alpha)}$. We easily have $f(z) \in \mathcal{F}_{n+p-1}(\alpha)$ and
\[
\text{Re} \left[ z^p D^{n+p-1} f(z) \right]^{\beta} \rightarrow 2^2 p (1-\alpha) \quad (2.36)
\]
as $z = \text{Re}(z) \to -1$. The proof of the corollary is complete. \qed

3. Convolution conditions. In [7], Silverman, Silvia, and Telage considered some convolution conditions for starlikeness of analytic functions. Recently, Silverman and Silvia [6] showed many necessary and sufficient conditions in terms of convolution operators for an analytic function to be in classes of starlike and convex. In this section, we give some necessary and sufficient conditions in terms of convolution operators for meromorphic functions to be in $\mathcal{F}_0^*(\alpha)$ and $\mathcal{F}_{n+p-1}(\alpha)$.

**Lemma 3.1.** Let $f(z) \in \sum_p$. Then $f \in \mathcal{F}_0^*(\alpha)$, $0 \leq \alpha < 1$, and $p \geq 1$, if and only if $f(z) * [(1-Az)/(zp(1-z)^2)] \neq 0$ ($0 < |z| < 1$), where
\[
A = \frac{1+x+2p(1-\alpha)}{2p(1-\alpha)}, \quad |x| = 1.
\] (3.1)

**Proof.** Let $f(z) \in \mathcal{F}_0^*(\alpha)$, then
\[
\text{Re} \left\{ \frac{-zf'(z)/f(z)-p\alpha}{p-p\alpha} \right\} > 0, \quad (3.2)
\]
which is equivalent to
\[
\frac{zf'(z)/f(z)+p\alpha}{p-p\alpha} \neq \frac{1-x}{1+x}, \quad |x| = 1, x \neq -1.
\] (3.3)

This simplifies to
\[
(zf'(z)+p\alpha f(z))(1+x)-(p-p\alpha)(1-x)f(z) \neq 0 \quad (3.4)
\]
in $0 < |z| < 1$. Using (2.5), we have
\[
f(z) * \frac{1}{z^p(1-z)^2} = zf'(z) + (1+p)f(z),
\] (3.5)
\[
f(z) * \frac{1}{z^p(1-z)} = f(z).
\]
Therefore, (3.4) is equivalent to
\[
f(z) * \left\{ \frac{1}{z^p(1-z)^2} - \frac{1+p}{z^p(1-z)} + \frac{p\alpha}{z^p(1-z)} \right\} (1+x) - (p-p\alpha)(1-x) \frac{1}{z^p(1-z)} \neq 0,
\] (3.6)
ON A CLASS OF MEROMORPHIC $p$-VALENT STARLIKE FUNCTIONS

that is,

$$f(z) \ast \frac{1 - \left(1 + x + 2p(1 - \alpha)\right)z}{z^p(1 - z)^2} \neq 0. \quad (3.7)$$

This proves Lemma 3.1.

\begin{proof}

The function $f(z) \in \mathcal{T}_{n+p-1}(\alpha)$ if and only if

$$f(z) \ast \frac{1 + [(n + p)(1 - A) - 1]z}{z^p(1 - z)^{n+p+1}} \neq 0 \quad (3.8)$$

for $0 < |z| < 1$, $|x| = 1$, where $A$ is given by (3.1).

\textbf{Theorem 3.2.} Since $f(z) \in \mathcal{T}_{n+p-1}(\alpha)$ if and only if $D^p f \in \mathcal{T}_{n+p-1}(\alpha)$, an application of (1.6) to Lemma 3.1 yields

$$f(z) \ast \left( g(z) \ast \left( \frac{1}{z^p(1 - z)^2} - \frac{Az}{z^p(1 - z)^2} \right) \right) \neq 0, \quad (3.9)$$

where $g(z) = 1/z^p(1 - z)^{n+p}$. In view of (3.5), we may write

$$g(z) \ast \left( \frac{1}{z^p(1 - z)^2} - \frac{Az}{z^p(1 - z)^2} \right) = g(z) \ast \frac{1}{z^p(1 - z)^2} - A g(z) \ast \frac{z}{z^p(1 - z)^2}$$

$$= z g'(z) + (1 + p)g(z) - A(z g'(z) + pg(z))$$

$$= (1 - A)z g'(z) + (1 + p - Ap)g(z)$$

$$= (1 - A) \cdot \frac{-p + (n + 2p)z}{z^p(1 - z)^{n+p+1}}$$

$$+ (1 + p - Ap) \cdot \frac{1}{z^p(1 - z)^{n+p}}$$

$$= 1 + [(n + p)(1 - A) - 1]z \neq 0. \quad (3.10)$$

This completes the proof of the theorem.
\end{proof}

\textbf{References}


JIN-LIN LIU: DEPARTMENT OF MATHEMATICS, YANGZHOU UNIVERSITY, YANGZHOU 225002, JIANGSU, CHINA

SHIGEYOSHI OWA: DEPARTMENT OF MATHEMATICS, KINKI UNIVERSITY, HIGASHI-Osaka, OSAKA 577-8502, JAPAN

*E-mail address: owa@math.kindai.ac.jp*
Mathematical Problems in Engineering

Special Issue on
Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>February 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>May 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>August 1, 2009</td>
</tr>
</tbody>
</table>

Guest Editors

**José Roberto Castilho Piqueira**, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

**Elbert E. Neher Macau**, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; elbert@lac.inpe.br

**Celso Grebogi**, Department of Physics, King’s College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk

Hindawi Publishing Corporation
http://www.hindawi.com