REPRESENTATION OF CERTAIN CLASSES OF DISTRIBUTIVE LATTICES BY SECTIONS OF SHEAVES

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ABSTRACT. Epstein and Horn ([6]) proved that a Post algebra is always a P-algebra and in a P-algebra, prime ideals lie in disjoint maximal chains. In this paper it is shown that a P-algebra L is a Post algebra of order \( n \geq 2 \), if the prime ideals of L lie in disjoint maximal chains each with \( n-1 \) elements. The main tool used in this paper is that every bounded distributive lattice is isomorphic with the lattice of all global sections of a sheaf of bounded distributive lattices over a Boolean space. Also some properties of P-algebras are characterized in terms of the stalks.

KEY WORDS AND PHRASES. Post Algebra, P-algebra, B-algebra, Heyting Algebra, Stone Lattice, Boolean Space, Sheaf of Distributive Lattices Over a Boolean Space, Prime Ideals Lie in Disjoint Maximal Chains, Regular Chain Base.

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1. **Introduction.**

Epstein ([5]) proved that in a Post algebra of order \( n \geq 2 \) prime ideals lie in disjoint maximal chains each with \( n - 1 \) elements. He has also proved that if \( L \) is a finite distributive lattice and prime ideals of \( L \) lie in disjoint maximal chains each with \( n-1 \) elements, then \( L \) is a Post algebra of order \( n \). Epstein and Horn ([6]) have shown that a Post algebra is always a P-algebra and in a P-algebra prime ideals lie in disjoint maximal chains. It is the main theme of this paper that a P-algebra \( L \) is a Post algebra of order \( n \geq 2 \), if the prime ideals of \( L \) lie in disjoint maximal chains each with \( n-1 \) elements.

The main tool used in this paper is the fact that every bounded distributive lattice is isomorphic with the lattice of all global sections of a sheaf of bounded distributive lattices over a Boolean space ([15] and [9]). It is well known that a P-algebra is always a (double) Heyting algebra, a (double) L-algebra, a pseudocomplemented lattice, a Stone lattice and a normal lattice. We characterize these properties of P-algebras in detail in terms of the stalks of the corresponding sheaf. We give another characterization of Post algebras by regular chain bases.

Throughout this paper, by \( L \) we mean a (nontrivial) bounded distributive lattice \((L, \lor, \land, 0, 1)\) and \( B = B(L) \) the Boolean algebra of complemented elements of \( L \). For any \( a \in B \), we denote the complement of \( a \) by \( a' \). For any \( x, y \in L \), we denote the largest \( z \in L \) such that \( x \land z \leq y \) (if it exists) by \( x \to y \) and the largest element \( a \in B \) such that \( x \land a \leq y \) (if it exists) by \( x \Rightarrow y \). If, for every \( x, y \in L, x \to y \) (\( x \Rightarrow y \)) exists, then we say that \( L \) is a Heyting algebra (respectively B-algebra). Dually, we define \( x \supset y \) and \( x \Leftarrow y \) respectively. If in a Heyting algebra (B-algebra), \( (x \to y) \lor (y \to x) = 1 \) (\( (x \Rightarrow y) \lor (y \Rightarrow x) = 1 \)) for every \( x, y \in L \), then we say that \( L \) is an L-algebra.
(respectively BL-algebra). For any $x \in L$, if $x \rightarrow 0$ exists, then we say that $x$ has the pseudocomplement and we usually write $x^*$ for $x \rightarrow 0$. If $x^*$ exists for each $x \in L$, then we say that $L$ is pseudocomplemented. The dual of $L$ is denoted by $L^d$. If both $L$ and $L^d$ are Heyting algebras (B-algebras, L-algebras, BL-algebras), then we say that $L$ is a double Heyting algebra (respectively double B-algebra, double L-algebra, double BL-algebra). $L$ is said to be a P-algebra if $L$ is a BL-algebra. Epstein and Horn proved that $L$ is a P-algebra if and only if $L$ is a double L-algebra ([6], theorem 3.4). For the elementary properties of these types of lattices, we refer to ([2]) and ([6]).

By a sheaf of bounded distributive lattices we mean a triple $(\mathcal{J}, \pi, X)$ satisfying the following:

1) $\mathcal{J}$ and $X$ are topological spaces
2) $\pi : \mathcal{J} \rightarrow X$ is a local homeomorphism
3) Each $\pi^{-1}(p)$, $p \in X$ is a bounded distributive lattice;
4) the maps $(x,y) \mapsto x \vee y$ and $(x,y) \mapsto x \wedge y$ from $\mathcal{J} \times \mathcal{J} = \{(x,y) \in \mathcal{J} \times \mathcal{J} / \pi(x) = \pi(y)\}$ into $\mathcal{J}$ are continuous and
5) the maps $\hat{0} : p \mapsto 0(p)$ and $\hat{1} : p \mapsto 1(p)$ from $X \rightarrow Q$ are continuous, where $0(p)$ and $1(p)$ are the smallest and largest elements of $\pi^{-1}(p)$ respectively.

We call $\mathcal{J}$ the sheaf space $X$ the base space and $\pi$ the projection map. We write $\mathcal{J}_p$ for $\pi^{-1}(p)$, $p \in X$ and call $\mathcal{J}_p$ the stalk at $p$. By a (global) section of the sheaf $(\mathcal{J}, \pi, X)$ we mean a continuous map $\sigma : X \rightarrow \mathcal{J}$ such that $\pi \circ \sigma = \text{id}_X$.

For any sections $\sigma$ and $\tau$ we write $|(\sigma, \tau)|$ for the closed set $\{p \in X | \sigma(p) \neq \tau(p)\}$ and we call $|(\sigma, 0)|$ the support of $\sigma$ and write $|\sigma|$ for $|(\sigma, 0)|$. For the preliminary results on sheaf theory, we refer to the pioneering work of Hofmann ([8]).
By Spec $L$, we mean the (Stone) space $Y$ of all prime ideals of $L$ with the hull-kernel topology; i.e., the topology for which $\{Y_x | x \in L\}$ is a base, where for any $x \in L$, $Y_x = \{p \in \text{Spec } L/x \notin P \}$. Throughout this paper $X$ denotes Spec $B$ which is a Boolean space, i.e., a compact, Hausdorff and totally disconnected space. Since $a \mapsto X_a$ is a Boolean isomorphism of $B$ onto the Boolean algebra of all clopen subsets of $X$, we identify $a$ and $X_a$ and write simply $a$ for $X_a$. For any $p \in X$, $\mathcal{J}_p$ be the quotient lattice $L/\theta_p$ where $\theta_p$ is the congruence on $L$ given by

$$(x,y) \in \theta_p \iff x \wedge a = y \wedge a \text{ for some } a \in B-p,$$

and let $\mathcal{S}$ be the disjoint union of all $\mathcal{J}_p$, $p \in X$. For each $x \in L$, define $\hat{x} : X \rightarrow \mathcal{S}$ by $\hat{x}(p) = \theta_p(x)$, the congruence class of $\theta_p$ containing $x$. Topologize $\mathcal{S}$ with the largest topology such that each $\hat{x}$, $x \in L$, is continuous. Define $\pi : \mathcal{S} \rightarrow X$ by $\pi(s) = p$ if $a \in \mathcal{J}_p$. The following theorem is the main tool used in this paper and is due to Subrahmanyam ([15]) (see also [9]).

**THEOREM 1.1** $(\mathcal{S}, \pi, X)$ described above is a sheaf of bounded distributive lattices in which each stalk has exactly two complemented elements, viz., $0(p)$ and $1(p)$.

1.2 For any $a \in B$, $p \in X$, $\hat{a}(p) = 1(p)$ if $p \in a$ and $\hat{a}(p) = 0(p)$ if $p \notin a$.

1.3 For any $x, y \in L$ and $a \in B$, $\hat{x}/a = \hat{y}/a$ if and only if $x \wedge a = y \wedge a$.

1.4 $\hat{x} \mapsto \hat{x}$ is an isomorphism of $L$ onto the lattice $\Gamma(x, \mathcal{S})$ of all global sections of the sheaf $(\mathcal{S}, \pi, X)$. We identify $\hat{x}$ with $x$ and write simply $x$ for $\hat{x}$.

1.5 For any prime ideal $P$ of $L$, there exists a unique $p \in X$ such that $\{x(p)/x \in P\}$ is a prime ideal of $\mathcal{J}_p$. On the other hand if $Q$ is a prime ideal of $\mathcal{J}_p$, $p \in X$, then $\{x \in L/x(p) \in Q\}$ is a prime ideal of $L$. This
correspondence exhibits the set of all prime ideals of $L$ as the disjoint union of the sets of prime ideals of the stalks. Moreover, if $P$ and $Q$ are prime ideals of distinct stalks $\mathcal{F}_P$ and $\mathcal{F}_Q$, then $P$ and $Q$ are incomparable, when they are regarded as prime ideals of $L$.

Throughout this paper, by a stalk $\mathcal{F}_P$, $p \in X$, we mean the stalks of the sheaf $(\mathcal{F}, \pi, X)$ described above at $p$.

2. PSEUDOCOMPLEMENTED LATTICES.

It is well known that a bounded distributive lattice is a Heyting algebra if and only if it is relatively pseudocomplemented; i.e., each interval $[x, y]$, $x \leq y \in L$ is pseudocomplemented ([1]). Also the class of all distributive pseudocomplemented lattices and the class of all Heyting algebras are equationaly definable (see [11] and [11]), when we regard the pseudocomplementation and $(x,y) \mapsto (x + y)$ as unary and binary operations respectively in $L$. Thanks to the referee for suggesting a simpler proof of the following.

THEOREM 2.1. $L$ is pseudocomplemented if and only if each stalk $\mathcal{F}_P$, $p \in X$ is pseudocomplemented and the pseudocomplementation $x \mapsto x^*$ is continuous and in this case, the pseudocomplement of $x(p)$ in $\mathcal{F}_p$ is precisely $x^*(p)$ for all $x \in L$.

PROOF. Suppose $L$ is pseudocomplemented. Then it is easily seen that for all $x$ and $p$, $(x(p))^*_{\mathcal{F}_p}$ exists and is equal to $x^*(p)$. Then it is easy to show that the map $x \mapsto x^*$ is continuous. For the converse, if $x \in L$, the hypothesis implies that the map $f : x \mapsto \mathcal{J}$ defined by $f(p) = (x(p))^*$ is a global section of $\mathcal{J}$. Therefore, $f = \hat{y}$ for some $y$ and it is clear that $y = x^*$.

If $L$ is a Heyting algebra, then each $\theta_a$, $a \in B$, is compatible with the binary operation $(x, y) \mapsto (x + y)$. For, if $a \in B$ and $(x, y)$ and $(x_1, y_1) \in \theta$ then $(x + x_1) \land y \land a = (x + x_1) \land x \land a \leq x_1 \land a = y_1 \land a \leq y_1$, so that $(x + x_1) \land a \leq (y + y_1) \land a$. Similarly, we have $(y + y_1) \land a \leq (x + x_1) \land a$ and hence
(x + y, y + z) ∈ θ a. Hence the following theorem and its proof are analogous to the above.

**THEOREM 2.2.** L is a Heyting algebra if and only if each stalk ∫ₚ, p ∈ X is a Heyting algebra, and the operation (s,t) → (s → t) of ∫ Y into ∫ is continuous and in this case x(p) → y(p) in ∫ₚ, p ∈ X, is equal to (x → y)(p) for all x,y ∈ L.

3. **NORMAL LATTICES.**

**DEFINITION 3.1.** (Cornish [4]). L is said to be normal if any two distinct minimal prime ideals of L are comaximal and L is said to be relatively normal if each interval [x,y], x ≤ y ∈ L is normal.

For any x,y ∈ L, let (x,y)* L be the ideal {z ∈ L/ x ∧ z ≤ y} of L. For any x ∈ L, we write (x)* L for (x,0)* L. Cornish ([4]) proved that L is normal if and only if (x ∧ y)* L = (x)* L V (y)* L for all x,y ∈ L, and that L is relatively normal if and only if (x ∧ y,z)* L = (x,z)* L V (y,z)* L for all x,y,z ∈ L where V stands for the join operation in the lattice of all ideals of L.

**THEOREM 3.2.** (Speed [13]). A pseudocomplemented distributive lattice is normal if and only if it is a Stone lattice.

**THEOREM 3.3.** (Balbes and Horn [1]): A Heyting algebra is relatively normal if and only if it is an L-algebra.

**THEOREM 3.4.** L is normal if and only if each stalk ∫ₚ, p ∈ X, is normal.

**PROOF.** Suppose L is normal and p ∈ X. Let u,v ∈ ∫ₚ so that u = x(p) and v = y(p) for some x,y ∈ L. Clearly (u) ∫ₚ = (v) ∫ₚ = (u ∧ v) ∫ₚ. Let t(p) ∈ (u ∧ v)* ∩ T L. Since, (x ∧ y ∧ t)(p) = x(p) ∧ y(p) ∧ t(p) = 0(p) there exists a ∈ B-p such that x ∧ y ∧ t ∧ a = 0, so that t ∈ (x ∧ y ∧ a)* ∩ T L and hence t = t₁ V t₂ for some t₁ ∈ (x ∧ a)* ∩ T L and t₂ ∈ (y ∧ a)* ∩ T L. Now t(p) = t₁(p) V t₂(p), t₁(p) ∈ (u)* ∩ T L and t₂(p) ∈ (v)* ∩ T L. Hence ∫ₚ is normal. Conversely, suppose each stalk ∫ₚ, p ∈ X is normal. Let x,y ∈ L and
For each $p \in X$, since $z(p) \in (x(p) \land y(p))_\mathcal{P}^*$, there exists $a \in \mathcal{B}_p$, $t$ and $s \in L$, such that $a \land z = a \land (t \lor s)$, $t \land x \land a = s \land y \land a = 0$. By the compactness of $X$, it follows that there exists $a_1, a_2, \ldots, a_n \in \mathcal{B}$ and $t_1, t_2, \ldots, t_n, s_1, \ldots, s_n \in L$ such that

$$
\begin{align*}
&\forall \ i \leq n \ a_i = 1, \\
&\forall \ i \leq n \ (t_i \lor s_i), \\
&\forall \ t \lor (t_i \land a_i) = 0 = s_i \land y \land a_i. \\
&\text{Now, Put } \ \ t = \lor (t_i \land a_i) \text{ and } \\
&\forall \ i \leq n \ a_i = 1, \\
&\forall \ i \leq n \ (t_i \land a_i \land x) = 0 = \lor (s_i \land a_i \land y) = s \land y. \\
&\text{Hence } (x \land y)^*_L \subseteq (x)^*_L \lor (y)^*_L
\end{align*}
$$

and the otherside inclusion is obvious. Hence $L$ is normal.

The proof of the following theorem is analogous to that of the above.

**THEOREM 3.5.** $L$ is relatively normal if and only if each stalk $\mathcal{Y}_p$, $p \in X$, is relatively normal.

**DEFINITION 3.6.** (Speed [12]). $L$ is said to be a distributive $*$ lattice and denoted by $L \in \Delta^*$ if, for each $x \in L$, there exists $y \in L$ such that

$$(x)_L^{**} = \{u \in L \mid u \land v = 0 \text{ for every } v \in (x)^*_L\} = (y)_L^*.$$

Speed ([12]) proved that $L \in \Delta^*$ if and only if, for each $x \in L$, there exists $y \in L$, such that $x \land y = 0$ and $x \lor y$ is dense; i.e., $x \land y)^*_L = \{0\}$.  

**THEOREM 3.7.** $L \in \Delta^*$ if and only if (i) $\mathcal{Y}_p \in \Delta^*$ for each $p \in X$ and (ii) $\{p \in X \mid x(p) \text{ is dense in } \mathcal{Y}_p\}$ is open for each $x \in L$.

**PROOF.** Suppose $L \in \Delta^*$ and $x \in L$. There exists $y \in L$ such that $x \land y = 0$ and $x \lor y$ is dense in $L$. Let $p \in X$. Clearly, $x(p) \land y(p) = 0(p)$. Also, if $z \in L$, such that $((x(p) \lor y(p)) \land z(p) = 0(p)$, then $(x \land y) \land z \land a = 0$ for some $a \in \mathcal{B}_p$ and hence $z \land a = 0$, so that $z(p) = 0(p)$. Hence $x(p) \lor y(p)$ is dense in $\mathcal{Y}_p$. Therefore $\mathcal{Y}_p \in \Delta^*$. Now, suppose $x(p)$ is dense in $\mathcal{Y}_p$. It follows that $y(p) = 0(p)$ and hence there exists $a \in \mathcal{B}_p$ such that $y \land a = 0$.  

We claim that \( x(q) \) is dense in \( \mathcal{J}_q \) for all \( q \in a \). For, if \( q \in a \) and \( z(q) \in \mathcal{J}_q \), \( z \in L \), such that \( x(q) \land z(q) = 0(q) \), then there exists \( b \in B - q \) such that \( x \land z \land b = 0 \); so that \( (x \lor y) \land z \land a \land b = 0 \), and hence \( z \land a \land b = 0 \) and since \( a \land b \in B - q \), \( z(q) = 0(q) \). Conversely, suppose (i) and (ii) hold and \( x \in L \).

For each \( p \in X \), by (i) and (ii), there exists \( y \in L \) and \( a \in B - p \) such that \( x \land y \land a = 0 \) and \( (x \lor y)(q) \) is dense in \( \mathcal{J}_q \) for all \( q \in a \). By the usual compactness argument, there exists \( y_1, y_2, \ldots, y_n \in L \), \( a_1, \ldots, a_n \in B \) such that

\[
\bigwedge_{i=1}^{n} a_i = 1, \quad a_i \land a_j = 0 \text{ for } i \neq j, \quad x \land y_i \land a_i = 0 \quad \text{and} \quad (x \lor y_i)(p) \text{ is dense in } \mathcal{J}_p \text{ for all } p \in a_i.
\]

Now put \( y = \bigvee_{i=1}^{n} (y_i \land a_i) \). Then \( x \land y = 0 \) and \( x \lor y \) is dense in \( L \). For, \( (x \lor y) \land z = 0 \) for some \( z \in L \), then, for all \( p \in a_i \),

\[
0(p) = ((x \lor y) \land z)(p) = (x(p) \lor y(p)) \land z(p) \text{ and hence } z(p) = 0(p) \text{ for all } p \in a_i \text{ and therefore } z = 0. \quad \text{Hence } L \in \Delta^*.
\]

4. **STONE LATTICES.**

For any \( p \in X \), since the stalk \( \mathcal{J}_p \) has exactly two complemented elements, \( \mathcal{J}_p \) is a Stone lattice if and only if \( \mathcal{J}_p \) is dense (i.e., if \( x(p) \not\vdash o(p) \), then \( (x(p))^*_p = \{0(p)\} \)). Hence, by theorem 2.1, 3.2, and 3.4, we have the following.

**THEOREM 4.1.** Suppose \( L \) is pseudocomplemented. Then the following are equivalent.

(i) \( L \) is a Stone lattice

(ii) \( L \) is normal

(iii) Each stalk \( \mathcal{J}_p \), \( p \in X \), is a normal

(iv) Each stalk \( \mathcal{J}_p \), \( p \in X \), is a Stone lattice

(v) Each stalk \( \mathcal{J}_p \), \( p \in X \), is dense.

The following theorem is a consequence of theorem 2.2, 3.3 and 3.5.
THEOREM 4.2. Let $L$ be a Heyting algebra. Then the following are equivalent.

(i) $L$ is an $L$-algebra

(ii) $L$ is relatively normal

(iii) Each stalk $\bigvee_p$, $p \in X$, is relatively normal

(iv) Each stalk $\bigvee_p$, $p \in X$, is an $L$-algebra.

Since $L$ is an $L$-algebra if and only if it is relatively Stone lattice (Theorem 4.11 of [1]) (i.e., each interval is a Stone lattice) in view of theorem 4.1, one may suspect that if $L$ is an $L$-algebra, then each stalk is relatively dense and hence a chain. This is not true (see 4.4 below), though the converse is proved in the following.

THEOREM 4.3. If $L$ is a Heyting algebra and each stalk is a chain, then $L$ is an $L$-algebra.

PROOF. If each stalk is a chain, then by theorem 1.5, the prime ideals of $L$ lie in disjoint maximal chains and hence $L$ is relatively normal lattice and hence the theorem follows from theorem 2.3.

EXAMPLE 4.4. Let $B_4$ be the 4-element Boolean algebra and $A$ be the distributive lattice obtained by adjoining an external element to $B_4$ as the smallest element. Then $A$ is an $L$-algebra which is not a chain (Thanks to the referee).

Epstein and Horn ([6]) proved that $L$ is a Stone lattice if and only if $L^d$ is pseudosupplemented and $0 \iff x \wedge y = (0 \iff x) \wedge (0 \iff y)$ for all $x, y \in L$. Now, these two necessary and sufficient conditions for $L$ to be a Stone lattice can be viewed in terms of the stalks as follows.

THEOREM 4.5. $L^d$ is pseudosupplemented if and only if $|x|$ is open for each $x \in L$ and in this case $|x| = 0 \iff x$ for all $x \in L$.

PROOF. Follows from Lemma 5.2.
For any \( p \in X \), let \( (p) \) be the smallest ideal of \( L \) containing \( p \). The proof of the following theorem is simple.

**THEOREM 4.6.** For any \( p \in X \), the stalk \( \mathcal{S}_p \) is dense if and only if \( (p) \) is a prime ideal of \( L \).

It can be easily seen that each stalk \( \mathcal{S}_p \), \( p \in X \), is dense if and only if \( |x \land y| = |x| \cap |y| \) for all \( x, y \in L \). Hence from theorem 4.5 and 4.6 and lemma 2.9 of (\cite{7}), we have the following.

**THEOREM 4.7.** \( L \) is a Stone lattice if and only if \( |x| \) is open for all \( x \in L \) and each stalk \( \mathcal{S}_p \), \( p \in X \) is dense.

**REMARK 4.8.** Swamy and Rama Rao (\cite{10}) proved that a lattice \( L \) is a Stone lattice if and only if \( L \) is isomorphic to the lattice of all global sections of a sheaf of dense bounded distributive lattices over a Boolean space in which each section has open support (see also \cite{9}). It can be verified, that when \( L \) is a Stone lattice, then our sheaf \( (\mathcal{S}, \pi, X) \) coincides with the sheaf constructed in (\cite{10}).

5. **\( p \)-ALGEBRAS.**

The following results interpret \( B \)-algebras in sheaf theoretic terminology.

**LEMMA 5.1.** Let \( x, y \in L \). Then \( x \Rightarrow y \) exists in \( B \) if and only if
\[
\{ p \in X / x(p) \leq y(p) \}
\]
is closed and in this case \( x \Rightarrow y = \{ p \in X / x(p) \leq y(p) \} \).

**PROOF.** For any \( p \in X \), \( x(p) \leq y(p) \) if and only if there exists \( a \in B \) such that \( x \land a \leq y \). If \( x \Rightarrow y \) exists in \( B \), then, for any \( p \in X \), \( x(p) \leq y(p) \) if and only if \( p \in x \Rightarrow y \). Conversely, if \( \{ p \in X / x(p) \leq y(p) \} \) is closed, then there exists \( a \in B \) such that \( p \in a \) if and only if \( x(p) \leq y(p) \) for all \( p \in X \). Hence \( a = x \Rightarrow y \).

The proof of the following is easy.

**LEMMA 5.2.** For any \( x, y \in L \), \( |(x, y)| \) is open if and only if there exists a largest element \( a \) of \( B \) such that \( x \land a = y \land a \).
The following theorem is a consequence of the above lemmas.

**THEOREM 5.3.** The following are equivalent.

1) \( L \) is a dual \( B \)-algebra

2) For any \( x, y \in L \), \( \{ a \in B / x \vee a = y \vee a \} \) is a principal filter of \( B \).

3) For any \( x, y \in L \), \( \{ a \in B / x \wedge a = y \wedge a \} \) is a principal ideal of \( B \).

4) \( L \) is a \( B \)-algebra

5) \( \{ p \in X / x(p) \leq y(p) \} \) is closed for every \( x, y \in L \).

6) \( \big| (x, y) \big| \) is open for every \( x, y \in L \).

**THEOREM 5.4.** Suppose \( L \) is a \( B \)-algebra. Then the following are equivalent.

1) \( L \) is a \( P \)-algebra; i.e. \( L \) is a \( BL \)-algebra

2) Each stalk is a chain

3) For every \( x, y \in L \), there exists \( a \in B \) such that \( x \wedge a \leq y \) and \( y \wedge a' \leq x \).

4) For every \( x, y \in L \), there exists \( a \in B \) such that \( x \vee a \geq y \) and \( y \vee a' \geq x \).

**PROOF.** 2 \( \iff \) 3 is proved in ([15]) and 3 \( \iff \) 4 is clear. 1 \( \iff \) 2 follows from lemma 5.1.

### 6. POST ALGEBRAS.

The following definition is slightly different from that of Chang and Horn ([3]).

**DEFINITION 6.1.** By a generalized Post algebra, we mean the lattice \( C(Z, C) \) of all continuous maps of a Boolean space \( Z \) into a discrete bounded chain \( C \) where, the operations are pointwise.

**THEOREM 6.2.** The following are equivalent

1) \( L \) is a generalized Post algebra.

2) There exists a chain \( C \) in \( L \) such that the natural map \( c \mapsto c(p) : C \to \mathcal{J}_p \) is an isomorphism for all \( p \in X \).

3) There exists a chain \( C \) and, for each \( p \in X \), an order isomorphism \( \alpha_p : C \to \mathcal{J}_p \) such that for any \( c \in C \) and \( x \in L \), \( \{ p \in X / \alpha_p(c) = x(p) \} \) is open in \( X \).
PROOF. 1 \implies 2:

Let \( L = C(Z,D) \) where \( Z \) is a Boolean space and \( D \) is a discrete bounded chain. It is well known that a \( \ominus \) \( \chi_a \) is a Boolean isomorphism of the algebra of all clopen subsets of \( Z \) onto \( B \), the centre of \( L \), where \( \chi_a \) is the characteristic function on \( a \). We identify \( \chi_a \) with \( a \). Also the Stone space \( X \) is homeomorphic with \( Z \).

Let \( C \) be the set of all constant maps of \( Z \) into \( D \). For any \( d \in D \), let \( d \) denote the constant map which maps every element of \( Z \) onto \( d \). Then \( C \) is a chain in \( L \). Let \( p \in X \). Clearly, the natural map \( \emptyset_p : C \to J_p = L/\emptyset_p \) is a homomorphism.

If \( d_1, d_2 \in D \) such that \( \bar{d}_1(p) = \bar{d}_2(p) \) then \( \bar{d}_1 \wedge a = \bar{d}_2 \wedge a \) for some \( a \in B-p \) and hence \( d_1 = d_2 \). Now, let \( x \in L \). Then if \( p \in x^{-1}(d) \) for some \( d \in D \), since \( x : Z \to D \) is continuous, \( x^{-1}(d) \in B-p \) and since \( x \wedge x^{-1}(d) = \bar{d} \wedge x^{-1}(d) \), it follows that \( (x,\bar{d}) \in \emptyset_p \). Hence \( \emptyset_p \) is an isomorphism.

2 \implies 3: If \( C \) is a chain in \( L \) and the natural map \( \emptyset_p : C \to J_p \) is an isomorphism for every \( p \in X \), then, for any \( c \in C \) and \( x \in L \) \( \{p \in X / \alpha_p(c) = x(p)\} \) which is open.

3 \implies 1: We first observe that since \( J_p \) is bounded and \( \alpha_p \) is an isomorphism of \( C \) onto \( J_p \), \( C \) is also bounded. Let \( X = \text{Spec } B \). Define \( \emptyset : L \to C(X,C) \) by \( (\emptyset(x))(p) = \alpha_p^{-1}(x(p)) \) for each \( x \in L \) and \( p \in X \). Let \( c \in C \). Then

\[
(\emptyset(x))^{-1}(c) = \{p \in X / \alpha_p^{-1}(x(p)) = c\}
\]

\[
= \{p \in X / \alpha_p(c) = x(p)\}
\]

is open by (3) and hence \( \emptyset(x) \) is continuous. Clearly \( \emptyset \) is a homomorphism and one-one since \( \alpha_p^{-1} \) is so. Now, we will show that \( \emptyset \) is onto. Let \( f \in C(X,C) \). Define \( \sigma : X \to J \) by \( \sigma(p) = \alpha_p(f(p)) \) for every \( p \in X \). We will show that \( \sigma \) is a section. Let \( x \in L \) and \( a \in B \), then
\[ \sigma^{-1}(x(a)) = \{ p \in a \mid \alpha_p(f(p)) = x(p) \} \]

\[ = \bigcap_{c \in C} \bigcup \{ p \in X \mid f(p) = c \} \cap \{ p \in X \mid \alpha_p(c) = x(p) \}. \]

Since \( f \) is continuous, it follows that \( \sigma^{-1}(x(a)) \) is open. Since \( \{ x(a) / a \in B \) and \( x \in L \) is a base for the topology on \( X \), it follows that \( \sigma \) is continuous and clearly \( \pi \circ \sigma = \text{id}_X \). Therefore, \( \sigma = x \) for some \( x \in L \) and also \( \theta(x) = f \).

Hence \( \theta \) is an isomorphism and therefore \( L \) is a generalized Post algebra.

**THEOREM 6.3.** Let \( n \geq 2 \) and \( L \) a \( P \)-algebra. Then the following are equivalent.

1) \( L \) is a Post algebra of order \( n \).

2) \( \text{Spec } L \) is the disjoint union of maximal chains each with \( n-1 \) elements.

3) Each stalk is a chain with \( n \) elements.

**PROOF.** 1 \( \Rightarrow \) 2 is proved in (\[5\]).

Since \( L \) is a \( P \)-algebra, by theorem 4.4, each stalk \( \bigcup_p \), \( p \in X \), a chain. Also, by theorem 0.5, \( \text{Spec } L \) is the disjoint union of the chains \( \text{Spec } \bigcup_p \), \( p \in X \).

If \( \text{Spec } L \) is the disjoint union of all maximal chains \( C_\alpha \), \( \alpha \in \Delta \) each with \( n-1 \) elements, then, for any \( p \in X \), \( \text{Spec } \bigcup_p = C_\alpha \) for some \( \alpha \in \Delta \). Hence \( \text{Spec } \bigcup_p \) has \( n-1 \) elements and therefore \( \bigcup_p \) has \( n \) elements and hence (2) \( \Rightarrow \) (3).

Now, suppose each stalk is a chain with \( n \) elements and \( C_n \) is the \( n \)-element chain \( 1 < 2 < \ldots < n \). For any \( p \in X \), let \( \bigcup_p = \{ 0(p) = x_{1p}(p) < x_{2p}(p) < \ldots < x_{np}(p) = 1(p) \} \) where \( x_{1p}, x_{2p}, \ldots, x_{np} \in L \). Define for any \( p \in X \),

\[ \alpha_p : C_n \rightarrow X_p \) by \( \alpha_p(i) = x_{ip}(p) \) for each \( i \in C_n \). Clearly, \( \alpha_p \) is an order isomorphism. Let \( i \in C_n \), \( x \in L \) and \( p \in X \) such that \( \alpha_p(i) = x(p) \). i.e., \( x_{ip}(p) = x(p) \) so that there exists \( a \in B_p \) such that \( x_{ip}(q) = x(q) \) for all \( q \in a \). Since \( L \) is a \( B \)-algebra and \( x_{jp}(p) < x_{kp}(p) \) for all \( j < k \), by theorem 5.3, there exists \( b \in B_p \) such that \( x_{jp}(q) < x_{kp}(q) \) for all \( j < k \) and \( q \in b \) and hence \( x_{ip}(q) = x_{iq}(q) \) for all \( i \in C_n \) and \( q \in b \). Then \( p \in a \wedge b \in B \) and
for any $q \in a \land b$, $\alpha(q) = x_{1q}(q) = x_{1p}(q) = x(q)$. Hence $\{p \in X / \alpha(p) = x(p)\}$ is open for each $i \in C$ and $x \in L$.

**DEFINITION 6.4.** By a chain base $C$ for $L$ we mean a chain $C$ with 0 in $L$ such that $L$ is generated by the centre $B$ and $C$; i.e., every $x \in L$ can be written in the form $\bigvee_{i=1}^{n} (a_{i} \land c_{i})$ for some $a_{i} \in B$ and $c_{i} \in C$.

**DEFINITION 6.5.** A chain base $C$ in $L$ is said to be regular, if, for $c_{1} \neq c_{2} \in C$ and $a \in B$, $c_{1} < c_{2}$ and $a \land c_{2} \leq c_{1}$ imply $a = 0$.

It is proved in ([15]) that a bounded distributive lattice $L$ is a generalized Post algebra if and only if there exists a regular chain base for $L$. Now, we characterize chain bases and regular chain bases in terms of the stalks. It is also proved in ([15]) that if $C$ is a chain base for $L$, the natural map $\emptyset : C \rightarrow \bigvee_{p} \emptyset(p)$, defined by $\emptyset : c(p) = c(p)$ is an epimorphism for all $p \in X$. We prove the converse in the following.

**THEOREM 6.6.** Let $C$ be a chain in $L$ and $0 \in C$. Then $\emptyset : C \rightarrow \bigvee_{p} \emptyset(p)$ is an epimorphism for each $p \in X$, if and only if $C$ is a chain base for $L$.

**PROOF.** Suppose $\emptyset : C \rightarrow \bigvee_{p} \emptyset(p)$ is an epimorphism for each $p \in X$ and let $x \in L$. For each $p \in X$, there exists $c_{p} \in C$ such that $\emptyset(p) = x(p)$ i.e., $c_{p}(p) = x(p)$, so that there exists $a \in B-p$ such that $c_{p} \land a = x \land a$. Therefore, there exists a partition $a_{1}, a_{2}, \ldots, a_{n}$ of $B$ and $c_{1}, c_{2}, \ldots, c_{n} \in C$ such that $c_{1} \land a_{1} = x \land a_{1}$ so that $x = x \land 1 = x \land \bigvee_{i=1}^{n} a_{i} = \bigvee_{i=1}^{n} (x \land a_{i}) = \bigvee_{i=1}^{n} (c_{1} \land a_{1})$. Hence $C$ is a chain base for $L$.

The following theorem is a straightforward verification.

**THEOREM 6.7.** Let $C$ be a chain in $L$. Then the following are equivalent.

1) The natural map $\emptyset : C \rightarrow \bigvee_{p} \emptyset(p)$ is one for all $p \in X$.

2) For any $c_{1} \neq c_{2} \in C$ and $a \in B$, $c_{1} < c_{2}$ and $a \land c_{2} \leq c_{1}$ imply $a = 0$. 


3) For any \( c_1 \neq c_2 \in C \) and \( 0 \neq a \in B \), \( a \land c_1 \neq a \land c_2 \).

By summarizing the above results, we have the following:

**THEOREM 6.8.** Suppose \( L \) is a bounded distributive lattice. Then the following are equivalent.

1) \( L \) is a generalized Post algebra

2) There exists a chain \( C \) in \( L \) such that the natural map \( \emptyset : C \rightarrow \bigvee_p \) is an isomorphism for all \( p \in X \).

3) There exists a chain \( C \) and for each \( p \in X \), an order isomorphism \( \alpha_p : C \rightarrow \bigvee_p \) such that for any \( c \in C \) and \( x \in L \), \( \{ p \in X \mid \alpha_p(c) = x(p) \} \) is open in \( X \).

4) \( L \) has a regular chain base.

**REMARK.** The equivalence of (1) and (4) is established in ([15]) by using the Boolean extension techniques.

**REFERENCES**


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