ABSTRACT. The Schur group of a commutative ring, $R$, with identity consists of all classes in the Brauer group of $R$ which contain a homomorphic image of a group ring $RG$ for some finite group $G$. It is the purpose of this article to continue an investigation of this group which was introduced in earlier work as a natural generalization of the Schur group of a field. We generalize certain facts pertaining to the latter, among which are results on extensions of automorphisms and decomposition of central simple algebras into a product of cyclics. Finally we introduce the Schur exponent of a ring which equals the well-known Schur index in the global or local field case.

KEY WORDS AND PHRASES. Schur index, group ring, automorphism.
AMS Subject classification 1979:
Primary 16A26

1. INTRODUCTION.
All rings discussed herein will be assumed to be commutative with 1. By an Azumaya $R$-algebra $A$ we will mean that $A$ is separable over $R$, faithful as an $R$-module and that $R:1$ coincides with the center of $A$. The symbol $B(R)$ will denote the Brauer group of classes of Azumaya $R$-algebras. We refer the reader to De Meyer et al [1] for fundamental facts pertaining thereto.

The symbol $S(R)$ will refer to those classes $[A]$ in $B(R)$ for which there exists a finite group $G$ and an $R$-algebra epimorphism from the group algebra $RG$ onto $A$. $S(R)$ is a subgroup of $B(R)$ called the Schur group of $R$. In De Meyer-Mollin [2] $S(R)$ was introduced and studied as a generalization of the Schur group of a field which has been extensively studied. As with the field case $S(R)$ was shown to be trivial when $R$ has non-zero characteristic. Therefore we assume henceforth that all rings have zero characteristic. It is worth noting by contrast that it was also proved in De Meyer-Mollin [2] that any finite Abelian group is the Schur group of some ring.

The purpose of this paper is to further study $S(R)$. We require certain notation and concepts, however, before proceeding.

Let $S/R$ be a Galois extension of rings with group $G = G(S/R)$. We define a crossed product algebra made with $S,R$ and a 2-cocycle (factor set) $\beta: G \times G \to U(S)$, units of $S$, by $(S/R,\beta) = \sum_{g \in G} S u_g$. This is the Azumaya $R$-algebra which is free as
as $S$-module on the basis $\{u_\sigma\}_{\sigma \in G}$ subject to the multiplication given by:

$$u_\sigma u_\tau = a^{c_\sigma(\sigma, \tau)}$$

where $a, b \in S$ and $\sigma, \tau \in G$.

If $G = \langle \sigma \rangle$ is cyclic of order $n$ then $(S/R, \alpha) = \sum_{i=1}^{n} S u_\sigma^n$ denotes the cyclic crossed product algebra in which $u^i_\sigma = \begin{cases} u & \text{if } 1 \leq i < n \\ a & \text{if } i = n. \end{cases}$

We refer the reader to De Meyer et al \[1\] for further details.

Now let $R$ be a connected ring; i.e. a ring whose only idempotents are $0$ and $1$. We define a cyclotomic crossed product algebra as one of the form $(\mathbb{F}[e(n)]/R, \beta)$ where $e(n)$ is a primitive $n$th root of unity which lies in the separable closure of $R$ and the values of $\beta$ are in the cyclic group generated by $e(n)$. If $R$ has finitely many idempotents then a cyclotomic $R$-algebra is the direct sum of cyclotomic algebras over the connected components of $R$. In De Meyer-Mollin \[2\] it was shown that the classes of $S(R)$ which contain a cyclotomic algebra form a subgroup which we denote by $S'(R)$.

The symbol $S''(R)$ will denote the subset of $S(R)$ consisting of those classes containing a homomorphic image of a separable group algebra $RG$. In De Meyer-Mollin \[2\] it was shown that if $R$ is an integrally closed Noetherian domain then $S''(R) \subseteq S'(R)$.

We conclude this section with some notation. $S(R, G)$ for a ring $R$ and a finite group $G$ will denote the subset of $S(R)$ consisting of those classes containing a homomorphic image of a separable group algebra $RG$. Although $S(R, G)$ will serve mainly as a notational device herein, it is worth noting that, when $R$ is a field, Spiegel and Trojan \[3\] have investigated $S''(R, G)$ from the point of view of determining when it is a group. $S'(R, G)$ and $S''(R, G)$ are defined in an analogous fashion to that of $S(R, G)$. Finally $S'(R, G) = S'(R) \cap S(R, G)$ and $S''(R, G) = S''(R) \cap S(R, G)$.

2. AUTOMORPHISMS AND SUBGROUPS OF AZUMAYA ALGEBRAS.

The first result generalizes certain facts about the Schur group of an abelian number field to the Schur group of a Dedekind domain.

The first part of the theorem is a generalization of a standard fact (see Yamada \[4\]). The second part is a generalization of an "Amitsur-type" question; i.e. a question concerning subgroups of certain Azumayas algebras; which is an extension of a question related to Amitsur's classification of subgroups of division rings (see Amitsur \[5\]). In what follows $\exp[A]$ denotes the exponent of $[A]$ in $B(R)$. Recall that all rings discussed herein have characteristic zero.

THEOREM 2.1.

Let $G$ be a finite multiplicative group and let $R$ be a Dedekind domain. Suppose that $[A] \in S'(R, G)$. Then:

1. $\exp[A]$, divides $|G|$, and
2. if $|G|$ is odd, $A$ has no nontrivial idempotents and $R$ has no nontrivial odd order roots of unity then $[A]$ is trivial and $G$ is cyclic.

PROOF:

Suppose that $K$ is the quotient field of $R$. Let $[A_R K] = [A(\chi, K)]$ where $A(\chi, K)$ denotes the simple component of $K$ corresponding to the absolutely irreducible character $\chi$ of $G$. Then by Herstein \[6, Theorem 4.4.5, p. 119\] we have that $\exp[A(\chi, K)]$
divides \( m_\chi \), the Schur index of \( \chi \) over \( K \), which in turn divides \(|G|\) by Curtis et al [7, Theorem 27.11, p. 585]. Since \( B(R) \to B(K) \) is one-to-one by De Meyer et al [1, Lemma 2.2, p. 136] then it follows that \( \exp[A] \) divides \(|G|\). This secures (1).

From De Meyer-Mollin [2] we have that if \( \exp[A] = n \) then \( e(n) \) is in \( R \). Hence \( S(R) = S(R)^2 \) the Sylow 2-subgroup of \( S(R) \), since \( R \) contains no nontrivial odd order roots of unity. However \(|G|\) is odd which means \([A] = [R]\) by (1). Since \( A \) has no nontrivial idempotents then \( A \) has minimal rank in its class in \( B(R) \); i.e. \( A = R \). But there is an R-algebra epimorphism \( \phi: RG \to A \). Hence \( \phi(G) \) is abelian. By Elgethun [8], \( \phi(G) \) is a subgroup of of a division ring which implies that all Sylow subgroups of \( G \) are cyclic. by Amitsur [5]. Therefore \( G \) is cyclic. This completes the proof of theorem 1.

The above result generalizes Mollin [9, Theorem 3.6, p. 243] to the ring case under the restriction that \( G \) generates \( A \). As pointed out in Elgethun [8], this restriction is necessary since subalgebras of arbitrary central separable algebras need not be separable over their centers.

In De Meyer-Mollin [2] it was proved that \( S(R) \) (and \( S'(R) \) when \( R \) has finitely many idempotents) are invariant subgroups of \( B(R) \) under the natural action of \( \text{Aut}(R) \) on \( B(R) \). We describe this action at the algebra level since we need it for the following results which give information pertaining to extensions of automorphisms. If \( [A] \in B(R) \) and \( \sigma \in \text{Aut}(R) \) then let \( \sigma A \) be the \( R \)-algebra equal to \( A \) as a rng but with \( R \)-algebra action given by the rule \( r * a = \sigma^{-1}(r) a \) for \( r \in R \) and \( a \in A \) with multiplication on the right being that in \( A \).

**Theorem 2.2.**

Let \( T \) be a connected ring which is a finite Galois extension of a local ring \( R \), with Galois group \( G = G(T/R) \). Suppose \([B] \in B(T)\) and \( B \) is maximally embedded in some \( R \)-Azumaya algebra \( A \) (i.e. \( B = C_A(T) = \text{centralizer of } T \text{ in } A \)). Then \( B \) is \( G \)-normal (i.e. every \( \sigma \in G \) extends to \( \text{Aut}(B) \)).

**Proof.**

By Childs [10, Theorem 5.1, p. 13] we have \([B] = [C_A(S)] = [A^\sigma \otimes R S]\), where \( A^\sigma \) is the opposite algebra of \( A \). Now by Childs et al [11, Theorem 1.2, p. 26] we have that \( \sigma \in G \) extends to an inner automorphism of \( A^\sigma \). Hence \([A^\sigma \otimes R S] = [A^\sigma \otimes R S]|_{\sigma} = f_\sigma(A^\sigma \otimes R S)\). Thus \([B] = [C_A]\) which implies that \( B \) is \( G \)-normal by De Meyer [12]. This secures theorem 2.2.

Maintaining the hypothesis of the theorem we have the following result which is immediate from theorem 2 and De Meyer [12, Theorem 12, p. 335].

**Corollary 2.3.**

\( B \cong C \otimes_R T \) for some \([C] \in B(R)\) if and only if \( B \) is the kernel of the Teichmuller cocycle map, (See Childs [10]).

Corollary 2.3 leaves open the following question. Given \([B] \in S(T)\), when is \([C] \in S(R)\)? For fields and quaternion algebras this question was attacked in Mollin [13], where we asked the specific question: When are all quaternion algebras in the Schur group of an imaginary subfield of a cyclotomic field 'induced' from the Schur group of its maximal real subfield. It is worth noting that it is well known that
such quaternion algebras are induced from quaternion algebras in the Brauer group of the maximal real subfield.

We now turn to the question: If \([A] \in S'(R)\) then when does \(\sigma \in Aut(R)\) extend to \(Aut(A)\)?

In De Meyer-Mollin [2] it was proved that when \(R\) is an integral domain then an element of order \(n\) in \(S'(R)\) is fixed by an element \(\sigma \in Aut(R)\) if and only if \(R\) contains \(e(n)\) fixed by \(\sigma\). When \(R\) is a field of characteristic zero we can obtain more; namely that the following are equivalent (see Mollin [14] and [9])

(i) \(\sigma\) fixes \(e(n)\)
(ii) \(\sigma\) extends to \(Aut(A)\)
(iii) \([A] = [\sigma A]\).

However the equivalence of (ii) and (iii) fails for rings in general. For rings, however, we do have De Meyer [12, Lemma 1, p. 328] which gives us the equivalence of:

(ii) and (iv): \(A\) is \(R\)-isomorphic to \(A\). Moreover an example in (De Meyer [12, p. 336] of a ring \(R\) is given to demonstrate that we may have (iii) without (iv). Nevertheless we do have the following result.

**PROPOSITION 2.4.**

Let \(R\) be a regular local ring or a ring of polynomials in one variable over a perfect field. If \([A] \in S'(R)\) and \(A\) has exponent \(n\) then the following are equivalent.

(1) \(R\) contains \(e(n)\) fixed by \(\sigma \in Aut(R)\)
(2) \(\sigma \in Aut(R)\) extends to \(Aut(A)\).
(3) \(\sigma A\) is \(R\)-isomorphic to \(A\).
(4) \([\sigma A] = [A]\).

**PROOF**

As remarked in the discussion preceding the proposition, (2) and (3) are equivalent. Moreover, (3) and (4) are equivalent by De Meyer [12, Proposition 3, p. 335]. Finally (1) and (4) are equivalent by De Meyer-Mollin [2], thereby securing the proposition.

**COROLLARY 2.5.**

Suppose \(T/R\) is an extension of regular local rings with Galois group \(G\). Suppose that \([B] \in S'(T)\) with \(\exp[B] = n\) and \(B\) is maximally embedded in some \(R\)-Azumaya algebra. Then \(e(n)\) is in \(R\).

**PROOF**

First we note that \(e(n)\) is in \(T\) by De Meyer-Mollin [2]. By theorem 2 every \(\sigma \in G\) extends to \(Aut(B)\). Hence by proposition 4 every \(\sigma \in G\) fixes \(e(n)\); i.e. \(e(n)\) is in \(R\). This completes the corollary.

3. **DECOMPOSITION OF AZUMAYA ALGEBRAS.**

One of the most important open questions in the theory of the Brauer group of a field \(R\) is: Is \(B(R)\) generated by cyclics? i.e. Are there integers \(m\) and \(n\) and cyclic algebras \(A_1, \ldots, A_r\) such that for an \(R\)-division algebra \(D\) with \([D] \in B(R)\) we have \(M_m(D) \cong M_n(R) \circ R A_1 \circ R \ldots \circ R A_r\) where \(M_*(\ast)\) denotes a full ring of \(m \times m\) matrices over \(*\). Recently, however Merkurjev and Suslin [15] were able to prove that if \(D\) has exponent \(t\), say, and \(e(t)\) is in \(R\), then \(D\) is a product of cyclic algebras of exponent \(t\), and in particular \(D\) has an abelian splitting field. This
result has special importance for $S(R)$ since we always have $e(t)$ in $R$ when there is an element of exponent $t$ in $S(R)$; i.e. the general question is answered by [15] for the Schur group of a field. Now, this does not mean that $D$ is isomorphic to a tensor product of cyclics. This was shown to be false in the early 1930's when A.A. Albert produced an example of a non-cyclic division ring of index and exponent 4. Now we turn to the ring case and attack similar questions.

For the remainder of this section we assume that $S$ is a connected ring, finite Galois over $R$ with Abelian group $G = G(S/R)$, and such that the Picard group of $S$ is trivial. Thus we have $G \cong \langle \sigma_1 \rangle \times \cdots \times \langle \sigma_t \rangle$ where the cyclic group $\langle \sigma_i \rangle$ has order $n_i$, say, for $i = 1, 2, \ldots, t$. Let $H_i = \prod_{j \neq i} \langle \sigma_j \rangle = G(S/R_i)$; i.e. $R_i$ is the fixed ring of $H_i$.

Now let $B(S/R)$ be the kernel of the map $B(R) \to B(S)$ given by $[A] \mapsto [A]_R$. By De Meyer et al [1, Corollary 1.3, p. 116] we have $B(S/R) \cong H^2(G, U(S))$, (the second cohomology group). Suppose now that $[A] \in B(S/R)$; then $A = (S/R, \beta)$ for some 2-cocycle $\beta$. A natural question to ask is: When does $A$ have a similar decomposition to that of $G$; i.e. as a tensor product of cyclic algebras $A_i$ where $[A_i] \in B(R_i/R)$ for $i = 1, 2, \ldots, t$? The following result provides an answer. We maintain the above notation and assumptions, and define a symmetric factor set on a group $G$ to be one such that $(\sigma, \tau) = (\sigma, \tau)$ for all $\sigma, \tau \in G$.

**THEOREM 3.1.**

Suppose that $G$ has order $n$ and exponent $m$ such that $n \in U(R)$ and $e(m) \in R$. If $[A] \in B(S/R)$ with $A = (S/R, \beta)$ then $A \cong A_1 \otimes_R \cdots \otimes_R A_t$ such that $[A_i] \in B(R_i/R)$ if $\beta$ is a symmetric factor set. Conversely we have a weaker result; viz: If $[A] = [A_1 \otimes_R \cdots \otimes_R A_t]$ with $[A_i] \in B(R_i/R)$ then $\beta$ cohomologous to a symmetric factor set.

**PROOF**

The proof follows exactly as in Mollin [16, Theorem 3.5] mutatis mutandis.

We note that the above generalizes the result for generic abelian crossed products over fields obtained in Amitsur et al [17, Lemma 1.5, p. 81].

Now maintaining the notation of the theorem we have the following result which yields a criterion for a decomposition of Schur group elements into a product of cyclics.

**COROLLARY 3.2.**

Let $R$ be an integrally closed Noetherian domain. Suppose that $[A] \in S''(R, C)$ where $\exp[A] = \exp G = m$. Then $[A] = [A_1 \otimes_R \cdots \otimes_R A_t]$ if and only if $\beta$ is cohomologous to a symmetric factor set. (Note that if $\beta$ is a symmetric factor set then we have the stronger result that $A \cong A_1 \otimes_R \cdots \otimes_R A_t$.)

**PROOF**

Since $RG$ is separable then the generalized Maschke theorem dictates that $|G| \in U(R)$. Since $R$ is an integrally closed Noetherian domain then $S''(R) \subseteq S'(R)$ from De Meyer-Mollin [2]. Moreover whenever there are elements of exponent $m$ in $S'(R)$ then by De Meyer-Mollin [2] we have that $e(m)$ is in $R$. The result now follows from Theorem 3.1.

The hypothesis of Corollary 3.2 which requires $\exp[A] = |G(S/R)|$ where $A \neq (S/R, \beta)$ does not always hold of course. In fact the question pertaining to when
such an $S$ exists is related (in the case where $R$ is a field) to the Brauer splitting
theorem, which states that a finite group of exponent $m$ has $\mathbb{Q}(e(m))$ as a splitting
field where $Q$ is the field of rational numbers. In Szeto [18] this theorem has been
generalized to the ring case, viz: If $G$ is a finite group of exponent $m$, and $R$ is a
connected ring with $RG$ being separable then $R[e(m)]$ is a splitting ring for $G$; i.e.
$R[e(m)]G$ is a direct sum of Azumaya algebras which are trivial in $B(R[e(m)])$.

In the case where $R$ is an abelian extension of $Q$ we investigated in Mollin [19]
and [20] conditions for the existence of a splitting field $S$ of $A$ where
$[A] \in S(R)$, $|S:R| = \exp[A]$ and $S \subseteq \mathbb{Q}(e_m)$ where $m = \exp G$ with $A$ being a simple component
of $RG$. An open area of inquiry is to obtain conditions for the existence of
such an $S$ when $R$ is a connected ring, in view of the generalized Brauer theorem cited
above. Finally this generalized splitting theorem allows us to formulate an analogue
of the Schur index over a ring. The next section deals with this development.

4. THE SCHUR EXPONENT OVER A RING.

When $R$ is a field of characteristic zero and $[A] \in S(R)$ then $A$ is a simple
component of $RG$ for some finite group $G$. Moreover, if $X$ is an absolutely irreducible
character of $G$ then $A(X,R) \cong M_n(D)$, a full ring of $n \times n$ matrices over a division ring
$D$ and, $\sqrt{|D:R|} = m_R(R)$ is called the Schur index of $X$ over $R$. For an arbitrary ring of
characteristic zero we do not have such a beautiful setup. However when we restrict
to a connected ring and use the generalized Brauer splitting theorem discussed in §3
then we do have a fundamental layout which closely approximates the above.

Suppose $S$ is a connected ring and $S$ is a splitting ring for a finite group $G$;
i.e. $SG = B_1 \oplus \cdots \oplus B_5$ then $\exp[B_i]$ is a sum of central separable $S$-algebras
whose classes are trivial in $B(S)$. In Szeto [18] and [21] characters $T^{(i)}$ of $G$ are
defined in such a way that they are in a one-to-one correspondence with the $B_i$.
Moreover it is established that $T^{(i)}(g)$ is a sum of $n_i$ roots of unity where $g \in G,
\exp g = 1$, and $S[T^{(i)}]$ is finitely generated, projective and separable over $S$.

Now, suppose that $R$ is a connected ring which does not split $G$, but $|G| \in U(R)$.
Then $RG = A_1 \oplus \cdots \oplus A_5$ where the $A_i$ are Azumaya over their centers. If $|G| = n$
and $R[e(n)]G = B_1 \oplus \cdots \oplus B_5$ then let $T^{(i)}$ be the character of $G$ corresponding to $B_i$.
Moreover it is established that $T^{(i)}(g)$ is a sum of $n_i$ roots of unity where $g \in G,
\exp g = 1$, and $S[T^{(i)}]$ is finitely generated, projective and separable over $S$.

Now we assume that $R$ is a regular domain. Therefore $B(R[e(n)]) \to B(K[e(n)])$ is
injective. Hence for $\sigma \in H_1$ we have $A[e(n)]$ have $[A[e(n)]] \to K[e(n)]$ $[e(n)] \to K[e(n)]$.
But $A_1 \otimes K = A(X_1,K)$ and by Dornhoff [23, Lemma 24.7, p. 124] $A(X_1,K) \otimes K[e(n)] \to \Sigma K[e(n)]$ $[e(n)]$.

where the sum ranges over all $\sigma \in G(K(X_1))$. Thus we have $A_1 \otimes C_1 R[e(n)] =$
\[\Sigma R[e(n)] \otimes C_1, \text{where } C_1 \text{ is the center of } A_1.\]
Therefore we have demonstrated that $\sigma \in H_1$
each $A_1$ corresponds exactly to one $T^{(i)}$. Hence we may define the Azumaya algebra
$A(T^{(i)},R)$ corresponding to $T^{(i)}$. Let $m_R(T^{(i)})$ be the exponent of $A(T^{(i)},R)$ in $S^n(C_1)$.
Thus we define for a regular domain $R$ the Schur exponent of $T^{(i)}$ over $R$; viz.
$m_R(T^{(i)})$. We note that this is a natural definition since in the case of a local or
a global field the exponent equals the index of the representative division algebra, i.e. the Schur index. We note that since \( R \) is a regular domain then \( R \) is an integrally closed Noetherian domain. Therefore by De Meyer-Mollin [2] we have

\[
[A(T(i), R)] \leq S''(C_1) \subset S'(C_1),
\]

i.e. as in the field case we are dealing with cyclotomic algebras.

We note that by theorem 2.1(1) we have that if \( R \) is Dedekind domain then \( m_R(T^{(1)}) \) divides \( |C| \) for \( [A(T^{(1)}, R)] \in S(C_1, C) \). This generalizes the standard fact for the Schur group of a field. We leave the reader with the fact that it is an open question as to whether or not other standard facts pertaining to the Schur index generalize (see Yamada, [4]).

**Acknowledgement**

This research is supported by N.S.E.R.C. of Canada.

**Reference**


Call for Papers

Space dynamics is a very general title that can accommodate a long list of activities. This kind of research started with the study of the motion of the stars and the planets back to the origin of astronomy, and nowadays it has a large list of topics. It is possible to make a division in two main categories: astronomy and astrodynamics. By astronomy, we can relate topics that deal with the motion of the planets, natural satellites, comets, and so forth. Many important topics of research nowadays are related to those subjects. By astrodynamics, we mean topics related to spaceflight dynamics.

It means topics where a satellite, a rocket, or any kind of man-made object is travelling in space governed by the gravitational forces of celestial bodies and/or forces generated by propulsion systems that are available in those objects. Many topics are related to orbit determination, propagation, and orbital maneuvers related to those spacecrafts. Several other topics that are related to this subject are numerical methods, nonlinear dynamics, chaos, and control.

The main objective of this Special Issue is to publish topics that are under study in one of those lines. The idea is to get the most recent researches and published them in a very short time, so we can give a step in order to help scientists and engineers that work in this field to be aware of actual research. All the published papers have to be peer reviewed, but in a fast and accurate way so that the topics are not outdated by the large speed that the information flows nowadays.

Before submission authors should carefully read over the journal's Author Guidelines, which are located at http://www.hindawi.com/journals/mpe/guidelines.html. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>July 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>October 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>January 1, 2010</td>
</tr>
</tbody>
</table>

Lead Guest Editor

Antonio F. Bertachini A. Prado, Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; prado@dem.inpe.br

Guest Editors

Maria Cecilia Zanardi, São Paulo State University (UNESP), Guaratinguetá, 12516-410 São Paulo, Brazil; cecilia@feg.unesp.br

Tadashi Yokoyama, Universidade Estadual Paulista (UNESP), Rio Claro, 13506-900 São Paulo, Brazil; tadashi@rc.unesp.br

Silvia Maria Giuliani Winter, São Paulo State University (UNESP), Guaratinguetá, 12516-410 São Paulo, Brazil; silvia@feg.unesp.br