A considerable number of equivalent formulas defining conditional value-at-risk and expected shortfall are gathered together. Then we present a simple method to bound the conditional value-at-risk of compound Poisson loss distributions under incomplete information about its severity distribution, which is assumed to have a known finite range, mean, and variance. This important class of nonnormal loss distributions finds applications in actuarial science, where it is able to model the aggregate claims of an insurance-risk business.

1. Introduction

Value-at-risk, or VaR for short, which is defined as the $\alpha$-quantile of a loss distribution for some prescribed confidence level $\alpha \in (0,1)$, is a popular measure of risk used to assess capital requirements in the insurance and finance industry. However, VaR suffers from various shortcomings pointed out in recent studies. For example, numerical instability and difficulties occur for nonnormal loss distributions, especially in the presence of “fat tails” and/or empirical discreteness. Furthermore, VaR is not a coherent measure of risk in the sense of Artzner et al. [6, 7], and it does not take into account the severity of an incurred adverse loss event.

A simple alternative measure of risk with some significant advantages over VaR is conditional value-at-risk or expected shortfall, abbreviated CVaR and ES, respectively, which is intuitively grasped as “the average of the 100(1 − $\alpha$)% worst losses.” This measure of risk is able to quantify dangers beyond VaR and it is coherent. Moreover, it provides a numerical efficient and stable tool in optimization problems under uncertainty. Some
recent studies presenting these advantages and further desirable properties include Acerbi [1], Acerbi et al. [2], Acerbi and Tasche [3, 4], Bertsimas et al. [8], Hürlimann [16, 17], Kusuoka [21], Pflug [26], Rockafellar and Uryasev [28, 29], Testuri and Uryasev [31], Wirch and Hardy [34], Yamai and Yoshiha [35, 36, 37].

The present paper gathers together a considerable number of equivalent formulas defining CVaR and ES, which are scattered through the recent literature on the subject. Beside this, it provides a simple method to bound the CVaR of compound Poisson loss distributions under incomplete information about its severity distribution. The latter is assumed to have a known finite range and a given mean and variance. This important class of nonnormal loss distributions finds applications in actuarial practice, where it is able to model the aggregate claims of an insurance risk business.

In Section 2, CVaR and ES are defined and a lot of their equivalent formulas are summarized. Furthermore, it is recalled that this measure of risk preserves the stop-loss order or, equivalently, the increasing convex order. Then, in Section 3, we show how to compute CVaR bounds for compound Poisson distributions knowing only the finite range, mean, and variance of the severity distribution. Finally, Section 4 contains a numerical illustration. It compares the average value of the obtained bounds with a normal approximation. The approximation turns out to be useful for large values of the Poisson parameter, where the bounds are difficult to evaluate numerically due to the underflow and overflow technical problem inherent in any computer-based quantitative evaluation.

2. Equivalent definitions and the stop-loss order-preserving property

Let \((\Omega, A, P)\) be a probability space such that \(\Omega\) is the sample space, \(A\) is the \(\sigma\)-field of events, and \(P\) is the probability measure. For a measurable real-valued random variable \(X\) on this probability space, that is, a map \(X : \Omega \to R\), the probability distribution of \(X\) is defined and denoted by \(F_X(x) = P(X \leq x)\).

In the present paper, \(X\) represents a loss random variable such that for \(\omega \in \Omega\), the real number \(X(\omega)\) is the realization of a loss-and-profit function with \(X(\omega) \geq 0\) for a loss and \(X(\omega) < 0\) for a profit. Given \(X\), consider the VaR to the confidence level \(\alpha\), defined as the lower \(\alpha\)-quantile,

\[
\text{VaR}_\alpha[X] = Q_X^h(\alpha) = \inf \{x : F_X(x) \geq \alpha\},
\]

and the upper conditional value-at-risk (CVaR\(^+\)) to the confidence level \(\alpha\), defined by Rockafellar and Uryasev [29] as the mean excess loss
above VaR,

\[ \text{CVaR}_\alpha^+[X] = E[X \mid X > \text{VaR}_\alpha[X]] \]  \hspace{1cm} (2.2)

The VaR quantity represents the maximum possible loss, which is not exceeded with the probability \( \alpha \) (in practice, \( \alpha = 95\%, \ 99\%, \ 99.75\% \)). The CVaR\(^+\) quantity is the conditional-expected loss given that the loss strictly exceeds its VaR. Next, consider the \( \alpha\)-tail transform \( X^\alpha \) of \( X \) with the distribution function

\[ F_{X^\alpha}(x) = \begin{cases} 0, & x < \text{VaR}_\alpha[X], \\ \frac{F_X(x) - \alpha}{1 - \alpha}, & x \geq \text{VaR}_\alpha[X]. \end{cases} \]  \hspace{1cm} (2.3)

Rockafellar and Uryasev [29] define CVaR to the confidence level \( \alpha \) as expected value of the \( \alpha\)-tail transform, that is, by

\[ \text{CVaR}_\alpha[X] = E[X^\alpha]. \]  \hspace{1cm} (2.4)

The obtained measure is a coherent risk measure in the sense of Artzner et al. [6, 7] and coincides with CVaR\(^+\) only under technical conditions, for example, in the case of continuous distributions (see Remarks 2.2 and Corollary 2.3). However, the reader should be warned that in many of the cited papers, the notion of CVaR is defined as the noncoherent measure (2.2) and is nevertheless claimed to be coherent. For instance, Pflug [26] proves the coherence of CVaR using (2.14), but then extends the proof to the noncoherent expression (2.2). Rockafellar and Uryasev [28] rely on Pflug’s result and bear the same mistake. Bertsimas et al. [8] define CVaR as (2.2), but then identify it with (2.5). In fact, all literature before spring 2001 defines CVaR as (2.2) and claims erroneously that it is coherent. It is only after the appearance of Acerbi et al. [2, 4] that Rockafellar and Uryasev [29] propose a clear distinction between the notions of CVaR\(^+\) and CVaR.

Alternatively, the ES to the confidence level \( \alpha \) is defined as

\[ \text{ES}_\alpha[X] = \frac{1}{\varepsilon} \cdot \int_{\alpha}^{1} \text{VaR}_u[X] \, du \]  \hspace{1cm} (2.5)

and represents the average of the 100\( \varepsilon\)% worst losses, where \( \varepsilon = 1 - \alpha \) denotes the loss probability. The CVaR and ES quantities coincide and satisfy a lot of equivalent formulas. The alternative expressions are based on several transforms associated with \( X \), which are of common use in the fields of reliability, actuarial science, finance, and economics.
The following standard definitions and notations are used throughout. The \textit{survival function} associated with the probability distribution of \(X\) is denoted by \(\overline{F}_X(x) = 1 - F_X(x)\). For \(u \in (0, 1)\), the \textit{upper} \(u\)-\textit{quantile} is the quantity \(Q_X^u(u) = \inf\{x : F_X(x) > u\}\), and an arbitrary \(u\)-\textit{quantile} \(Q_X(u)\) denotes an element of the interval \([Q_X^0(u), Q_X^u(u)]\). The \textit{stop-loss transform} of \(X\) is the real-valued function defined by \(\pi_X(x) = E[(X - x)_+] = \int_x^\infty \overline{F}_X(t)\, dt\), where \(x_+ = x\) if \(x \geq 0\) and \(x_+ = 0\), otherwise. The \textit{mean excess function} of \(X\) is the real-valued function defined by \(m_X(x) = E[X - x | X > x] = \pi_X(x)/\overline{F}_X(x)\). Under the \textit{right-spread transform} of \(X\), we mean the real-valued function defined by \(S_X(u) = \pi_X(Q_X(u)), u \in [0, 1]\) (e.g., Fernandez-Ponce et al. [11] and Shaked and Shanthikumar [30]). The \textit{Lorenz transform} of \(X\) is defined by \(L_X(u) = \int_0^u Q_X(t)\, dt, u \in [0, 1]\), while the \textit{dual Lorenz transform} is \(L_X^*(u) = \int_u^1 Q_X(t)\, dt\). A standard reference for the Lorenz transform is Arnold [5], while its dual has been considered by Heilmann [12]. Another important probability transform is the \textit{Hardy-Littlewood transform} defined by \(HL_X(u) = L_X^*(u)/(1 - u)\) if \(u \in [0, 1]\) and \(HL_X(1) = Q_X(1)\), which has been considered in many papers (e.g., Kertz and Rösler [18, 19, 20], Hürlimann [15], and references therein). We know that it identifies with the quantile function \(Q_{XHL}(u) = HL_X(u), u \in [0, 1]\), of a random variable \(X^{HL}\) associated with \(X\), which is called here \textit{Hardy-Littlewood random variable} and which turns out to be the least majorant with respect to the stochastic dominance of first order among all random variables \(Y\) preceding \(X\) in the increasing convex order (e.g., Meilijson and Nádas [22]). For an increasing concave function \(g : [0, 1) \rightarrow [0, 1]\) such that \(g(0) = 0, g(1) = 1\), we consider, in actuarial science, the \textit{distortion transform} of \(X\) defined by \(D_X^g[X] = \int_0^1 (g[\overline{F}_X(t)] - 1)\, dt + \int_0^\infty g[\overline{F}_X(t)]\, dt\) (e.g., Wang et al. [33] and references herein). Finally, the \textit{total-time-on-test} or \textit{TTT-transform} of \(X\) is the real-valued function defined by \(T_X(u) = \int_0^{Q_X(u)} \overline{F}_X(t)\, dt, u \in (0, 1)\), and is widely used in reliability. We note that many properties of the transforms \(L_X, T_X\), and their relationships have been discussed in Pham and Türkkan [27].

\textbf{Proposition 2.1.} Let \(X\) be a real-valued random variable defined on the probability space \((\Omega, A, P)\). Then \(\text{CVaR}_\alpha[X] = \text{ES}_\alpha[X]\), and these quantities can be represented by the following equivalent formulas:

\[
\frac{F_X(\text{VaR}_\alpha[X])}{1 - \alpha} \cdot \text{VaR}_\alpha[X] + \frac{1 - F_X(\text{VaR}_\alpha[X])}{1 - \alpha} \cdot \text{CVaR}_\alpha^+[X], \]

\[
Q_X(\alpha) + \frac{1}{\epsilon} \cdot S_X(\alpha),
\]

\[
\frac{1}{\epsilon} \cdot \{E[X] - L_X(\alpha)\},
\]
\[
\frac{1}{\varepsilon} \cdot L_X^\ast (\alpha),
\]
\[
HL_X (\alpha),
\]
\[
Q_X^\alpha (\alpha),
\]
\[
D_{g^\varepsilon} [X], 
\]
\[
g^\varepsilon (x) = \min \left\{ \frac{x}{\varepsilon}, 1 \right\},
\]
\[
\frac{1}{\varepsilon} \cdot \left\{ E [X \cdot 1_{\{X > Q_X (\alpha)\}}] + Q_X (\alpha) \cdot (\varepsilon - \bar{F_X} [Q_X (\alpha)]) \right\},
\]
\[
\min_{\xi} \left\{ \xi + \frac{1}{\varepsilon} \cdot \pi_X (\xi) \right\}.
\]

**Proof.** We first show that \( \text{CVaR}_\alpha [X] = \text{ES}_\alpha [X] \). Rearranging and making a change of variables, we obtain

\[
\text{ES}_\alpha [X] = Q_X^h (\alpha) + \frac{1}{\varepsilon} \cdot \int_0^1 (Q_X^h (u) - Q_X^h (\alpha)) \, du
\]
\[
= Q_X^h (\alpha) + \frac{1}{\varepsilon} \cdot \int_{Q_X (\alpha)}^{\infty} (x - Q_X^h (\alpha)) \, dF_X (x)
\]
\[
= Q_X^h (\alpha) + \frac{1}{\varepsilon} \cdot \pi_X [Q_X^h (\alpha)].
\]

On the other hand, by definition of CVaR, we have

\[
\text{CVaR}_\alpha [X] = E [X^\alpha] = \int_0^{\infty} \bar{F_X} (x) \, dx - \int_{-\infty}^0 F_X (x) \, dx.
\]

Using (2.3) and distinguishing between the two cases \( \text{VaR}_\alpha [X] \geq 0 \) and \( \text{VaR}_\alpha [X] < 0 \), we, without difficulty, obtain that

\[
\text{CVaR}_\alpha [X] = \text{VaR}_\alpha [X] + \frac{1}{\varepsilon} \cdot \pi_X [\text{VaR}_\alpha [X]] = \text{ES}_\alpha [X].
\]

The weighted average formula (2.6) is Proposition 6 in Rockafellar and Uryasev [29]. Since the integral in (2.5) does not depend on the choice of the \( \alpha \)-quantile, we, similarly to the above, obtain that

\[
HL_X (\alpha) = \frac{1}{\varepsilon} \cdot \int_0^1 Q_X (u) \, du = \text{ES}_\alpha [X] = Q_X (\alpha) + \frac{1}{\varepsilon} \cdot \pi_X [Q_X (\alpha)],
\]

which yields (2.7) and (2.10). Now, (2.9) is immediate by the definition of the Hardy-Littlewood transform, and (2.8) follows immediately through rearrangement, noting that \( E[X] = \int_0^1 Q_X (u) \, du \). The relationship (2.11) is clear by the definition of Hardy-Littlewood random variable.
Formula (2.12) is an easy exercise. Formula (2.13), which expressed in terms of the worth or gain random variable \( -X \) is (3.11) in Acerbi and Tasche [4], is obtained as follows. We have

\[
E[X \cdot 1_{\{|X > Q_X(\alpha)\}}] = E[(X - Q_X(\alpha)) \cdot 1_{\{|X > Q_X(\alpha)\}}] \\
+ E[Q_X(\alpha) \cdot 1_{\{|X > Q_X(\alpha)\}}] \\
= E[(X - Q_X(\alpha))] + Q_X(\alpha) \cdot \Phi_X[Q_X(\alpha)],
\]

which, inserted in (2.13), immediately yields (2.7). Finally, the minimization formula (2.14) is found in Rockafellar and Uryasev [29]. □

Remarks 2.2. Up to (2.6), (2.13) and (2.14) and under the assumption of continuous distributions, these equivalent expressions for CVaR are also derived in Hürlimann [17]. The many available alternative formulations for CVaR and ES suggest that, besides most recent ones, several proofs of the coherence of this measure are known. In particular, the distortion transform representation (2.12) can be traced back to Denneberg [9, 10], Wang [32], Wang et al. [33], and Hürlimann [15], which contain proofs of the coherence of this measure.

Besides the identification of CVaR\( ^+ \) with CVaR in the case of continuous distributions, we note that a huge number of further equivalent formulas could be found. Only two attractive possibilities are mentioned.

**Corollary 2.3.** Under the assumption of continuous distributions, CVaR\( ^+ \)[X] = CVaR\( \alpha \)[X] and these quantities are equivalent to the following formulas:

\[
Q_X(\alpha) + m_X[Q_X(\alpha)], \\
\frac{1}{\varepsilon} \cdot E[X] - \int_0^{1-\varepsilon} \frac{T_X(x)}{(1-x)^2} dx.
\]

**Proof.** If the distribution function is continuous, we have that \( F_X(VaR_\alpha[X]) = \alpha \) and (2.6) coincides with (2.2). Formula (2.20) follows from (2.7), noting that

\[
m_X[Q_X(\alpha)] = \frac{\pi_X[Q_X(\alpha)]}{\Phi_X[Q_X(\alpha)]} = \frac{1}{\varepsilon} \cdot S_X(\alpha).
\]

Finally, (2.21) follows from a result due to Pham and Turkkan [27, Theorem 2 and formula (5)]. We have\( T_X(u) = L_X(u) + (1-u) \cdot Q_X(u) \).
Since $Q_X(u)$ is continuous, we have $L'_X(u) = Q_X(u)$, which yields a linear differential equation in $L_X(u)$. Its solution is

$$L_X(u) = (1 - u) \cdot \int_0^u \frac{T_X(x)}{(1 - x)^2} dx.$$  \hfill (2.23)

Inserted in (2.8), we obtain (2.21). \hfill $\square$

In the special situation of discrete arithmetic loss distributions defined on the nonnegative integers, which will be used to evaluate our CVaR bounds in Section 3, numerical evaluation proceeds as follows. Let $f_k = \Pr(X = k)$ denote the probability that the nonnegative loss takes the value $k$, where $k = 0, 1, 2, \ldots$, and assume that the finite mean $\mu_X = E[X]$ is known. Determine the unique index $k_\alpha$ such that

$$\sum_{k=0}^{k_{\alpha} - 1} f_k < \alpha \leq \sum_{k=0}^{k_{\alpha}} f_k.$$  \hfill (2.24)

Then we have

$$\text{VaR}_\alpha[X] = Q_X^b(\alpha) = k_\alpha,$$  \hfill (2.25)

and we obtain from (2.7) that

$$\text{CVaR}_\alpha[X] = Q_X(\alpha) + \frac{1}{\varepsilon} \cdot \left\{ \mu_X - Q_X(\alpha) + E[(Q_X(\alpha) - X)_+] \right\}$$

$$= \frac{1}{\varepsilon} \cdot \left\{ \mu_X - \alpha \cdot k_\alpha + \sum_{k=0}^{k_{\alpha}} (k_\alpha - k) \cdot f_k \right\}.$$  \hfill (2.26)

In particular, the loss probabilities must only be evaluated up to the index $k_\alpha$ satisfying inequality (2.24).

It is important to observe that the CVaR functional is preserved under the stop-loss order or equivalently the increasing convex order. This fact is a main ingredient underlying the construction of CVaR bounds in Section 3. The next result is a slight generalization of Theorem 1.1 in Hürlimann [16], which is valid there for continuous distributions only. Recall that a loss $X$ precedes another one $Y$ in the stop-loss order, written $X \leq_{\text{sl}} Y$ if $\pi_X(x) \leq \pi_Y(x)$ for all $x$.

**Proposition 2.4.** Let $X$ and $Y$ be two real-valued random variables defined on the probability space $(\Omega, A, P)$. Then $X \leq_{\text{sl}} Y$ if and only if $\text{CVaR}_\alpha[X] \leq \text{CVaR}_\alpha[Y]$ for all $\alpha \in [0, 1]$. 
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Proof. By (2.11) we have $\text{CVaR}_\alpha[X] = Q_{X^{HL}}(\alpha)$ and $\text{CVaR}_\alpha[Y] = Q_{Y^{HL}}(\alpha)$. The result follows from the fact that $X \leq_{sl} Y$ if and only if $X^{HL} \leq_{sl} Y^{HL}$, where $\leq_{sl}$ denotes the stochastic dominance of first order (e.g., Kertz and Rösler [20, Lemma 1.8], or Hürlimann [15, Theorem 2.3]). □

3. CVaR bounds for compound Poisson risks

An important risk management issue of an insurance company is the construction of more or less accurate bounds on risk measures like CVaR or ES for compound random sums $S = X_1 + \cdots + X_N$, where the claim number $N$ is Poisson ($\lambda$), the claim sizes $X_i$ are independent and identically distributed as $X$, and $X_i$ is independent from $N$. By incomplete information about the claim size, say $X$ belongs to the set $D = D([0,b];\mu,\sigma)$ of all nonnegative random variables with maximum claim size $b$, known mean $\mu$, and standard deviation $\sigma$, simple bounds are obtained as follows.

Following Hürlimann [14, Section 3], consider the stop-loss-ordered extreme random variables $X_{\min}$ and $X_{\max}$ for the set $D$ such that

$$X_{\min} \leq_{sl} X \leq_{sl} X_{\max}, \quad \forall X \in D. \quad (3.1)$$

Then replace $X$ by $X_{\min}$ and $X_{\max}$ in the compound Poisson random sums to get random sums $S_{\min}$ and $S_{\max}$ such that

$$S_{\min} \leq_{sl} S \leq_{sl} S_{\max}, \quad \forall X \in D. \quad (3.2)$$

Since CVaR is preserved under stop-loss order by Proposition 2.4, we obtain the bounds

$$\text{CVaR}_\alpha[S_{\min}] \leq \text{CVaR}_\alpha[S] \leq \text{CVaR}_\alpha[S_{\max}], \quad \forall \alpha \in [0,1]. \quad (3.3)$$

For computational reasons, it is more advantageous to evaluate bounds based on finite atomic claim sizes. Now, the minimum $X_{\min}$ is already 2-atomic while the maximum $X_{\max}$ has a probability distribution of mixed discrete and continuous type. The latter can be replaced through mass dispersion by a 4-atomic stop-loss larger discrete approximation $X_{\max}^d$ such that $S_{\max} \leq_{sl} S_{\max}^d$ and $\text{CVaR}_\alpha[S_{\max}] \leq \text{CVaR}_\alpha[S_{\max}^d]$ for all $\alpha \in [0,1]$. Recall the structure of the finite atomic random variables $X_{\min}$ and $X_{\max}^d$. Let $v = (\sigma/\mu)^2$, the relative variance of the claim size, $v_0 = (b - \mu)/\mu$, the maximum relative variance for the set $D$, and $v_\tau = v/v_0$, a relative variance ratio. The discrete supports and probabilities of these
random variables are described for $X_{\min}$ by

$$\{x_1, x_2\} = \{(1 - v_r)\mu, (1 + v)\mu\}, \quad \{p_1, p_2\} = \left\{\frac{v_0}{1 + v_0}, \frac{1}{1 + v_0}\right\}, \quad (3.4)$$

and for $X_{\max}^d$ by

$$\{x_0, x_1, x_2, x_3\} = \left\{0, \frac{1}{2}(1 + v)\mu, \left[1 + \frac{1}{2}(v_0 - v_r)\right]\mu, (1 + v_0)\mu\right\},$$

$$\{p_0, p_1, p_2, p_3\} = \left\{\frac{v}{1 + v'}, \frac{v_0 - v}{(1 + v)(1 + v_0)}, \frac{v_0 - v}{(1 + v_0)}\right\}.$$  \quad (3.5)

In practice, we choose the parameters and fix a unit of money in such a way that the atoms $x_i$ are nonnegative integers. Recall that the probabilities $f_k$, $k = 0, 1, 2, \ldots$, of a compound Poisson ($\lambda$) distribution with non-negative integer claim sizes $x_0 = 0 < x_1 < \cdots < x_m$, and the corresponding probabilities $p_0, p_1, \ldots, p_m$ are best numerically evaluated using the following Adelson-Panjer recursive algorithm (e.g., Panjer [23], Hürlimann [13]):

$$f_0 = e^{-\lambda(1 - p_0)}, \quad f_k = \frac{\lambda}{k} \sum_{j=1}^{m} \delta(k - x_j)x_j p_j f_{k-x_j}, \quad k = 1, 2, 3, \ldots, \quad (3.6)$$

where $\delta(x) = 1$ if $x \geq 0$ and $\delta(x) = 0$ else. Finally, to obtain CVaR$_{\alpha}[S]$, we use formulas (2.24) and (2.26).

Since computers represent only a finite number of digits, it remains to discuss the technical problems of round-off errors and underflow/overflow. Regarding round-off errors, it has been shown by Panjer and Wang [24] that the recursive formula (3.6) is strongly stable such that this algorithm works well. However, for large values of $\lambda$, underflow/overflow occurs. In this situation, some methods have been proposed in Panjer and Willmot [25]. In Section 4, we use exponential scaling/descaling as follows. Let $\mu_S = \lambda \mu$, $\sigma^2_S = \lambda(\mu^2 + \sigma^2)$ be the mean and variance of $S$. Choose appropriately $M = \mu_S - t \cdot \sigma_S$ for some $t$ ($t = 19, 25.5$ in our example in Section 4 for $\lambda = 2000, 3000$), and let $r = \lambda(1 - p_0)/M$, $m_0 = [M]$, the greatest integer less than $M$. Exponential scaling and recursion yields

$$h_0 = 1, \quad h_k = \frac{\lambda}{k} \sum_{j=1}^{m} \delta(k - x_j)x_j p_j e^{-r x_j} h_{k-x_j}, \quad k = 1, 2, \ldots, m_0. \quad (3.7)$$

Then apply exponential descaling setting $f_k = h_k e^{r(k-M)}$, $k = 0, \ldots, m_0$, and continue the evaluation of $f_k$ for $k > m_0$ with the recursion (3.6).
4. Bounds on the insurance economic-risk capital

We are interested in the evaluation of economic-risk capital of an insurance portfolio whose compound Poisson aggregate claims $S$ at a future date are covered by a risk premium $P > \mu_S$. The future random loss of the portfolio can be decomposed as follows:

$$S - P = (\mu_S - P) + (S - \mu_S).$$

(4.1)

The first component, which is the negative of the insurance margin, represents the future expected insurance gain and belongs to the stakeholders of the insurance company. To protect this expected gain, we require some economic-risk capital to cover the insurance loss $L = S - \mu_S$ (signed deviation from the mean aggregate claims). Using CVaR as risk measure, the future value of this economic-risk capital is equal to

$$\text{CVaR}_\alpha[L] = \text{CVaR}_\alpha[S] - \mu_S,$$

(4.2)

where $\alpha$ is some prescribed confidence level. Note that the equality in (4.2) follows from the translation invariant property of CVaR, which is one of the axioms required to define a coherent risk measure.

The following numerical illustration is based on the approximate figures of a real-life portfolio of grouped life insurance contracts from the early 1980s. For some unit of money, our choice for the claim-size parameters is $\mu = 12$, $\sigma^2 = 360$, and $b = 48$; hence $\nu = 5/2$, $\nu_0 = 3$, and $\nu_r = 5/6$. According to (3.4) and (3.5), the discrete supports and probabilities are given for $X_{\min}$ by

$$\{x_1, x_2\} = \{2, 42\}, \quad \{p_1, p_2\} = \{0.75, 0.25\},$$

(4.3)

and for $X_{\min}^d$ by

$$\{x_0, x_1, x_2, x_3\} = \{0, 21, 25, 48\},$$

$$\{p_0, p_1, p_2, p_3\} = \{0.71429, 0.03571, 0.03261, 0.21739\}.$$

(4.4)

Table 4.1 displays the values $\text{CVaR}_{\min} = (1/\mu_S) \cdot \text{CVaR}_\alpha[L_{\min}]$ and $\text{CVaR}_{\max} = (1/\mu_S) \cdot \text{CVaR}_\alpha[L_{\max}^d]$ with $L_{\min} = S_{\min} - \mu_S$, $L_{\max}^d = S_{\max} - \mu_S$, which represent bounds on the insurance economic-risk capital per unit of mean aggregate claims for $\alpha = 95\%, 99\%, 99.75\%$ by varying the expected number of claims $\lambda$. The average rate

$$\text{CVaR}^A = \frac{1}{2} (\text{CVaR}_{\min} + \text{CVaR}_{\max})$$

(4.5)
Table 4.1. CVaR bounds and normal approximation as percentages of $\mu_S$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\lambda$</th>
<th>CVaR$_{\text{min}}$</th>
<th>CVaR$_{\text{max}}$</th>
<th>CVaR$^A$</th>
<th>CVaR$^N$</th>
<th>$D^N$</th>
</tr>
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<tr>
<td>95%</td>
<td>100</td>
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<td>41.944</td>
<td>40.033</td>
<td>38.590</td>
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is compared with the normal approximation rate

\[
\text{CVaR}^N = \frac{1}{\varepsilon} \varphi[\Phi^{-1}(\alpha)] \cdot \frac{\sigma_S}{\mu_S},
\]

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(1/2)t^2} \, dt, \quad \varphi(x) = \Phi'(x),
\]

which is obtained by approximating $S$ by a normal random variable $S^N$ with a mean $\mu_S$ and a standard deviation $\sigma_S$. The approximation error is measured here by the signed normal deviation rate

\[
D^N = \text{CVaR}^N - \text{CVaR}^A.
\]

The following observations are noted. By fixed confidence level $\alpha$, the normal approximation underestimates the average rate up to some fixed, rather large, expected number of claims $\lambda$, and then overestimates it. The underestimation increases by increasing the confidence level $\alpha$. Since
computational difficulties with the exponential scaling/descaling method of Section 3 arise for values of \( \lambda \) beyond 3000, the normal approximation appears useful in this range provided insurers agree to set insurance economic-risk capital rates at the proposed average rate (4.5).

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References


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E-mail address: whurlimann@bluewin.ch
Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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