SUMS OF SQUARES OF PURE QUATERNIONS

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Abstract

We show that for each $k \geq 1$ there is a division quaternion algebra $D$ of level $s(D) = 2^k$ such that $-1$ is not the sum of squares of $2^k$ pure quaternions in $D$. This answers a question asked by D. W. Lewis (Rocky Mountain Journal of Mathematics 19 (1989), 787–92).

The problem of determining the level of quaternion algebras was discussed by D. W. Lewis in [3] and D. B. Leep in [2]. The approach used by Lewis associates with a quaternion algebra $D = \left( \frac{a}{b} \right)$ the quadratic form $T_P = \langle a, b, -ab \rangle$ over the field $F$. Lewis has shown that, for any positive integer $n \in \mathbb{N}$, if $\langle 1 \rangle \perp nT_P$ is isotropic over $F$, then $-1$ is a sum of $n$ squares of quaternions in $D$ (see [3, lemma 4]). He commented on the converse of this implication and stated that it is true for $n = 2^k - 1$, $k \geq 2$, but for other values of $n$ we do not know.

In this note we show that, in general, this converse statement is not true. We construct an explicit example of a quaternion algebra $D$ over a formally real field $K$ with the property that for $n = 2^k$, the element $-1 \in D$ is a sum of $n$ squares in $D$, but the quadratic form $\langle 1 \rangle \perp nT_P$ is anisotropic over $K$. We also show that $-1$ cannot be expressed as a sum of $n-1$ squares in $D$.

We begin with a refinement of [3, lemma 4].

Lemma 1. Let $n$ be any positive integer, $D = \left( \frac{a}{b} \right)$ and $T_P = \langle a, b, -ab \rangle$. Then the quadratic form $\langle 1 \rangle \perp nT_P$ is isotropic over $F$ if and only if $-1$ can be expressed as a sum of $n$ squares of pure quaternions in $D$.

Proof. This is implicit in [3, lemma 4] and [2, theorem 2.2]. But for the sake of completeness we sketch a proof.

The isotropicity of $\langle 1 \rangle \perp nT_P$ over $F$ is equivalent to $nT_P$ representing $-1$ over $F$, that is, to the existence of $q_1, r_1, s_1, \ldots, q_n, r_n, s_n \in F$ such that

$$-1 = T_P(q_1, r_1, s_1) + \cdots + T_P(q_n, r_n, s_n).$$

Since for a pure quaternion $c = qi + rj + sk$, we have $c^2 = T_P(q, r, s)$, such a representation of $-1$ exists if and only if there are pure quaternions $a_m = q_m i + r_m j + s_m k$, $1 \leq m \leq n$, satisfying

$$-1 = a_1^2 + \cdots + a_n^2,$$

as desired. 

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