FIXED POINTS, MULTIVALUED INEQUALITIES, CONTROL PROBLEMS AND DIFFERENTIAL INCLUSIONS ON PROXIMATE RETRACTS

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ABSTRACT

Generalised multivalued inequalities are studied in this paper. Our results enable us to discuss control problems and differential inclusions on proximate retracts.

1. Introduction

This paper discusses the existence problem for generalised multivalued variational inequalities involving upper semicontinuous maps which are acyclic [10] or approximable [5], [19]. We divide the paper into three main sections. In §2 we gather together some known fixed point results [5], [10], [16], [17], [19]. Also a new fixed point result will be established. Section 3 will use the fixed point theory of §2 to establish some general variational inequalities. The results in §3 extend previously known results [6], [18], [19] in several directions. Finally in §4 we give applications of the results obtained in the previous sections. In particular we discuss control problems and differential inclusions on proximate retracts.

We now gather together some definitions and known facts. Let $E$ be a Hausdorff linear topological vector space and $Y$ be a topological space. A mapping $F: E \to 2^Y$ (here $2^Y$ denotes the family of all non-empty subsets of $Y$) is upper semicontinuous (u.s.c.) if the set $F^{-1}(B) = \{ x \in E : F(x) \cap B \neq \emptyset \}$ is closed for any closed set $B$ in $Y$. We recall the following well-known result for u.s.c. maps [21].

**Theorem 1.1.** Let $E$ and $Y$ be topological spaces and $F: E \to 2^Y$ a u.s.c. point-compact multifunction. Suppose that $\{ x_n \}$ is a net in $E$ with $y_n \in F(x_n)$ for each $n$. If $x_n \to x_0$ and $y_n \to y_0$, then $y_0 \in F(x_0)$.

Suppose that $X$ and $Z$ are subsets of Hausdorff topological vector spaces $E_1$ and $E_2$, respectively and $F: X \to 2^Z$ is a multifunction. Given two open neighbourhoods $U$ and $V$ of the origins in $E_1$ and $E_2$, respectively, a $(U, V)$-approximative continuous selection [4] of $F$ is a continuous function $s: X \to Z$ satisfying

$$s(x) \in (F[(x + U) \cap X] + V) \cap Z \quad \text{for every} \quad x \in X.$$  

$F$ is said to be approximable [5], [18] if its restriction $F|_K$ to any compact subset $K$ of $X$ admits a $(U, V)$-approximative continuous selection for every open neighbourhoods $U$ and $V$ of the origins in $E_1$ and $E_2$, respectively.