PERTURBATION AND COPERTURBATION FUNCTIONS
CHARACTERISING SEMI-FREDHOLM-TYPE OPERATORS

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Abstract
We introduce the concepts of a perturbation function and a coperturbation function, which allow us to give a general approach to the question of obtaining characterisations and perturbation theorems for $F_+$ and $F_-$ operators.

1. Introduction

Certain norm-related functions have been considered in order to obtain characterisations and perturbation results for various classes of semi-Fredholm-type operators.

In the §2, by means of what we have called a perturbation function, we present a general approach to the question of obtaining characterisations and perturbation theorems for $F_+$ and strictly singular operators. To this end, for each perturbation function $\psi$, we construct functions $\Gamma_\psi$ and $\Delta_\psi$, which are similar to $\Gamma$ and $\Delta$ introduced by Cross [1].

In §3, we study the $F_-$ operator and strictly cosingular operator, in a similar way to what we have done before for $F_+$ and strictly singular operators, but using the notion of a coperturbation function. For this purpose, we define, for each coperturbation function $\nu$, the functions $(\Gamma')$ and $(\Delta')$ which are similar to $\Gamma'$ and $\Delta'$ considered by Labuschagne [7] (see also [10]).

Let $X$ and $Y$ be infinite dimensional normed spaces. The collections of infinite dimensional, closed infinite co-dimension, finite dimensional, finite codimensional and closed finite co-dimension subspaces of $X$ are respectively denoted by $I(X), I_c(X), F(X), C(X)$ and $P(X)$.

$L(X,Y)$ denotes the class of linear operators defined on a subspace $D(T)$ of $X$ having range in $Y$. The range and null space of $T$ are respectively denoted by $R(T)$ and $N(T)$. For a subspace $M$ of $X$, $i_M$ denotes the element of $L(X,X)$ which is the canonical injection of $M$ into $X$, with the quotient map from $X$ onto $X/M$ being denoted by $q_M$. We write $J_X$ for the injection of $X$ into its completion $X^\sim$, and $X'$ is the dual of $X$. Let $J$ denote the operator in $L(D(T),X)$ that is the identity injection of $D(T)$ into $X$. Then the conjugate $T'$ of $T$ is defined to be the conjugate of $TJ$ in the sense of [5].

Square brackets will be used to indicate that only everywhere-defined operators are considered, for example, $PK[X,Y]$ denotes the class of everywhere-defined precompact operators in $L(X,Y)$. Continuous everywhere-defined operators will be referred to as bounded operators.

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An operator $T \in \mathcal{L}(X, Y)$ is said to be $F_\omega$ if $T$ is continuously invertible on a finite codimensional subspace of $D(T)$ [1], and $F_\omega$ if there exists $M \in F(Y)$ such that $(q_M T)'$ has a continuous inverse [7]. The classes of all $F_\omega$ and $F_\omega$ operators in $\mathcal{L}(X, Y)$ will be denoted by $F_c(X, Y)$ and $F_c(X, Y)$, respectively.

$T$ is defined to be strictly singular [2] (strictly cosingular [7]) if there is no $M \in I(D(T))(M \in I_c(Y))$ such that $T_{1_M}((q_M T)')$ has a continuous inverse. The class of strictly singular (strictly cosingular) operators in $\mathcal{L}(X, Y)$ will be denoted by $SS(X, Y)$ (SC$(X, Y)$).

We remark that this definition of strictly singular (strictly cosingular) generalises that of [6] ([13]) with the two definitions being equivalent in the classical case of bounded operators between Banach spaces [5, 4-4(11)]. We similarly conclude from [3, 6] and [4, 4-25 and 5-7] that $F_\omega$ and $F_\omega$ operators coincide with upper and lower semi-Fredholm operators, respectively, in the case of closed operators in Banach spaces.

The injection modulus of $T$, $j(T)$, is defined by $j(T) := \inf \{ \|Tx\| : \|x\| = 1 \}$ [14] and the minimum modulus of $T$ by $\gamma(T) := \sup \{ x : \|x\| \leq \|Tx\|, x \in D(T) \}$ [8] where $d(x, N(T))$ is the distance of $x$ from $N(T)$. We observe that $j(T) > 0$ if and only if $T$ has a continuous inverse, $\gamma(T) > 0$ if and only if $T$ is relatively open [4, 4.21].

For $T \in \mathcal{L}(X, Y)$, Cross [1] defined the quantities $\Gamma$, $\Gamma_\omega$, $\Gamma_\omega$, and $\Delta$ as follows: if $\dim D(T) < \infty$ then $\Gamma(T) = 0$ for all $f \in \{ \Gamma, \Gamma_\omega, \Gamma_\omega, \Delta \}$. If $\dim D(T) = \infty$ then $\Gamma(T) := \inf \{ \|T|M\| : M \in I(D(T)) \}$, $\Gamma_\omega(T) := \inf \{ \|T|M\| : M \in C(X) \}$, $\Gamma_\omega(T) := \inf \{ \|T|M\| : M \in P(X) \}$, and $\Delta(T) := \sup \{ \|T|M\| : M \in I(D(T)) \}$, and proved that if $\dim D(T) = \infty$ then $T \in F_c(X, Y)$ if and only if $\Gamma(T) > 0$, and $T \in SS(X, Y)$ if and only if $\Delta(T) = 0$.

Labuschagne [7] (see also [4] and [10]) defined $\Gamma'(T) := \inf \{ \|q_M J_M T\| : M \in I_c(Y \sim) \}$, $\Gamma_\omega'(T) := \inf \{ \|q_M J_M T\| : M \in F(Y) \}$, and $\Delta'(T) := \sup \{ \|q_M T\| : M \in I_c(Y) \}$, and proved that $T \in \mathcal{F}_\infty(X, Y)$ if and only if $\Gamma'(T) > 0$, and $T \in SC(X, Y)$ if and only if $\Delta'(T) = 0$.

We will also consider the measures of non-precompactness $\rho$, $\kappa$ and the measure of non-strict-singularity $SS$ (non-strict-cosingularity SC) which are defined by $\rho(T) := \inf \{ r : TB_{D(T)} \text{ is covered by a finite set of open balls in } Y \text{ of radius } r \}$ (here $B_{D(T)} := \{ x \in D(T) : \|x\| \leq 1 \}$), $\kappa(T) := \inf \{ \|T - S\| : S \in PK[X, Y] \}$, $SS(T) := \inf \{ \|T - S\| : S \in SS[X, Y] \}$ and $SC(T) := \inf \{ \|T - S\| : S \in SC[X, Y] \}$.

The quantity $\rho$ was introduced by Cross [1], $\kappa$ was considered by Lebow–Schechter [11], $SS$ and $SC$ have been defined by Rakocevic [15] and Martinon [12], respectively, but only in the context of bounded operators between Banach spaces.

2. Perturbation functions and $F_\omega$-operators

**Definition 2.1.** We define a perturbation function to be a function $\psi$ assigning to each pair of normed spaces $X, Y$ and $T \in \mathcal{L}(X, Y)$ a number $\psi(T) \in [0, \infty]$, verifying the following properties:

(i) $\psi(\lambda T) = |\lambda| \psi(T)$, $\lambda \in K$.
(ii) $\psi(T + S) = \psi(T)$, $S \in PK[X, Y]$.
(iii) $\psi(T) \leq \|T\|$.
2.8], from Lemma 2.2, we conclude that $c$ properties are clear.

Next, we show that the quantities $\rho$, $\kappa$, $\Gamma_O$, $\Gamma_O$, $\Delta$ and $SS$ are perturbation functions. We need the following lemma.

**Lemma 2.2.** Let $T \in L(X, Y)$. Then $(\Gamma_O)'(T) \leq \rho(T)$ with equality if $T$ is continuous.

**Proof.** Suppose $\rho(T) < \infty$ (i.e. $T$ continuous), since the assertion is trivial otherwise. By the method of proof of [12, theorem 23.1], it can be shown that $(\Gamma_O)'(T) = \rho(T)$.

In general, the inequality is strict. In fact, it suffices to consider a semi-continuous operator $T$ (or equivalently $(\Gamma_O)'(T) < \infty [7, 1.1(IV)]$ which is not continuous, e.g. $T$ a discontinuous finite rank operator.

**Proposition 2.3.** Let $f \in \{\rho, \kappa, \Gamma_O, \Gamma_O, \Delta, SS\}$. Then $f$ is a perturbation function.

**Proof.** $f = \rho$: it is known that $\rho$ has properties (i)–(iii) and (v) [1]. To prove (iv) we can assume without loss of generality that $T$ is continuous and $R(T) = Y$, so that, $N(T) = \{0\}$. Then by [4, 5.4] we have that $\gamma(T') \leq \Gamma(T)$ and since $\gamma(T) = \gamma(T')$ [8, 2.8], from Lemma 2.2, we conclude that $\gamma(T) \leq \rho(T)$.

$f = \kappa$: since $\rho(T) \leq \kappa(T) [1]$, as in the previous case, we obtain (iv) and the other properties are clear.

$f \in \{\Gamma_O, \Gamma_O\}$: property (v) is proved in [1, 2.17], while properties (i) and (iii) follow from the definitions. Let $S \in PK[X, Y]$, then by [1, 2.14 and 3.2], since $T = (T+S) - S$, we have $\Gamma(T) = \Gamma(T+S)$. Property (iv) is obtained by noting that if $\dim D(T) = \infty$ and $\dim N(T) < \infty$ then $\gamma(T) \leq \Gamma(T)$ [4, 5.4].

$f \in \{\Delta, SS\}$: from the definitions and [1, 2.18, 3.4 and 2.10] it follows immediately that $\Gamma(T) \leq \Delta(T) \leq SS(T)$ and that $f$ has properties (i)–(iii) and (v). Now (iv) is obtained as in the above case.

**Definition 2.4.** Given a perturbation function $\psi$, for $T \in L(X, Y)$, we define the functions $\Gamma_\psi$ and $\Delta_\psi$ as follows: If $\dim D(T) < \infty$ then $\Gamma_\psi(T) = 0$, while if $\dim D(T) = \infty$ then $\Gamma_\psi(T) := \inf \{\psi(T_i M): M \in \text{I}(D(T))\}$ and $\Delta_\psi(T) := \sup \{\psi(T_i M): M \in \text{I}(D(T))\}$.

**Proposition 2.5.** Let $T \in L(X, Y)$. Then $\Gamma_\psi(T) = \Gamma(T)$ and $\Delta_\psi(T) = \Delta(T)$ if $\psi \in \{\kappa, \Gamma_O, \Gamma_O\}, \Delta, SS\}$. 

**Proof.** By Proposition 2.3, $\psi$ is a perturbation function and so $\Gamma_\psi(T) \leq \Gamma(T)$. Since $\Gamma(T) \leq \Delta(T) \leq \Gamma_O(T) \leq \Gamma_O(T) \leq \kappa(T) [1]$ and $\Delta(T) \leq SS(T)$ we have that $\Gamma(T) \leq \Gamma(T_i M) \leq \psi(T_i M), M \in \text{I}(D(T))$ and consequently $\Gamma(T) \leq \Gamma_\psi(T)$. The second assertion now follows trivially.

Combining the equality $\Gamma(T) = \inf \{\Gamma_O(M): M \in \text{I}(D(T))\}$ with the property $\Gamma_O(T) \leq 2\rho(T) [1, 5.1]$, we obtain that $\Gamma_\rho(T) \leq \Gamma(T) \leq 2\Gamma_\rho(T)$ and $\Delta_\rho(T) \leq 2\Delta(T)$. 


In the following theorems we will always assume that $\psi$ is a perturbation function. Our next theorem is an unbounded analogue of [16, theorem 2.14]

**Theorem 2.6.** Let $T, S \in L(X, Y)$ such that $\Delta_\psi(S) < \Gamma_\psi(T)$. Then $T + S \in F_\psi(X, Y)$.

**Proof.** Suppose $T + S \notin F_\psi(X, Y)$, then there is $M \in I(D(T + S))$ such that $(T + S)M$ is precompact [1, 2.2]. Therefore, $\Gamma_\psi(T) \leq \Gamma_\psi(TM) = \Gamma_\psi(SM) \leq \Delta_\psi(S)$. This contradicts the hypothesis. ■

**Theorem 2.7.** Let $T \in L(X, Y)$. Then

(i) If $\dim D(T) = \infty$ then $T \in F_\psi(X, Y)$ if and only if $\Gamma_\psi(T) > 0$.

(ii) $T \in SS(X, Y)$ if and only if $\Delta_\psi(T) = 0$.

**Proof.** (i) If $\Gamma_\psi(T) > 0$, by taking $S = 0$ in the above theorem we find $T \in F_\psi(X, Y)$.

If $T \in F_\psi(X, Y)$ then there exist closed subspaces $M$ and $W$ with $W$ finite dimensional such that $M \oplus W \oplus N(T) = D(T)$ for which $TM$ has a continuous inverse [1, 2.2]. Then $0 < \gamma(TM)$ and if $V \in I(D(T))$ the subspace $H := M \cap V$ is also infinite dimensional and $0 < \|TM\| < \|TV\| = \gamma(TM) \leq \psi(TM) \leq \psi(TV)$ and taking the infimum over all $V \in I(D(T))$ we get $\Gamma_\psi(T) > 0$.

(ii) Since $\Delta_\psi(T) \leq \Delta(T)$ and $T \in SS(X, Y)$ if and only if $\Delta(T) = 0 [1, 3.4]$ it follows that $\Delta_\psi(T) = 0$ if $T \in SS(X, Y)$. To prove the other implication, assume that $T \notin SS(X, Y)$. Then there exists $M \in I(D(T))$ such that $\Gamma(TM) > 0$. Define $S \in L(X, Y)$ by $D(S) = M$ and $Sx = -Tx$, $x \in M$. Then $S \in F_\psi$ and $S + T = 0$. But if $\Delta_\psi(T) = 0$ then by (i) and the previous theorem we have that $S + T \notin F_\psi$. ■

### 3. Coperturbation functions and $F_\psi$-operators

In this section we introduce the notion of coperturbation function in order to deduce characterisations and perturbation theorems for $F_\psi$ and strictly cosingular operators.

**Definition 3.1.** A coperturbation function will be a function which determines, for each pair of normed spaces $X$, $Y$ and $T \in L(X, Y)$, a number $(T) \in [0, \infty]$ with the following properties:

(i) $(\lambda T) = |\lambda| (T), \lambda \in \mathbb{K}$.

(ii) $(T + S) = (T), S \in PK[X, Y]$.

(iii) $(T) \leq \|T\|$. (iv) $\gamma(T) \leq (T)$ whenever $\dim D(T) = \infty$ and $\dim N(T') < \infty$.

(v) $(q_M T) \leq (T), M \in I(Y)$.

**Proposition 3.2.** Let $f \in \mathcal{F}(\rho, \kappa, (\Gamma_\psi)^\prime, \Delta, SC)$. Then $f$ is a coperturbation function.

**Proof.** $f \in \mathcal{F}(\rho, \kappa)$; properties (i)–(iii) and (v) are known [1]. To verify (iv) let $\rho(T) < \infty$ and $\gamma(T') > 0$, since the inequality is trivial otherwise. Then, considering Lemma 2.2 and [4, 5.4], we get $\gamma(T') \leq \Gamma(T) \leq \Delta(T) \leq (\Gamma_\psi)^\prime(T) \leq \rho(T) \leq \kappa(T)$.

$f \in \mathcal{F}(\Gamma_\psi)^\prime, \Delta, SC$; properties (i), (iii) and (v) are established in [7, 2.1(II)]. Let $S \in PK[X, Y]$, then from Lemma 2.2, [7, 2.3(II) and 2.4(II)] and [4, 4.15], we
have \((\Gamma_0)'(T) \leq (\Gamma_0)'(T+S) + (\Gamma_0)'(S) = (\Gamma_0)'(T+S) + (\Gamma_0)'(S) + \Delta'(T+S) - \Delta'(T)\) \leq \Delta'(J_{Y}S)
+ \Delta'(T+S) \leq (\Gamma_0)(J_{Y}S) + \Delta'(T+S) \leq (\Gamma_0)(S) + \Delta'(T+S) = \Delta'(T+S) \leq \Delta'(T)\nand \(\text{SC}(T) = \text{SC}(T+S)\), i.e. \(f\) has property (iii). Moreover, \(\Delta'(T) \leq \text{SC}(T)\) since if \(S \in \text{SC}[X, Y]\) then \(\Delta'(T) \leq \Delta(S) + \Delta(J_{Y}(T+S)) \leq (\Gamma_0)(T+S) \leq \|T+S\|\). Now, the proof where the \(f\) satisfies (iv) is analogous to the above case.

**Definition 3.3.** Given a coperturbation function \(\Gamma\), for \(T \in L(X, Y)\) we define \((\Gamma')(T) := \inf\{ (q_M J_{Y} T): M \in I_0(Y\sim)\}\) and \((\Delta')(T) := \sup\{ (\Gamma')(q_M T): M \in I_0(Y)\}\).

**Proposition 3.4.** Let \(T \in L(X, Y)\). Then \((\Gamma')(T) = \Gamma'(T)\) and \((\Delta')(T) = \Delta'(T)\) if \(\varepsilon \in \rho(\kappa, (\Gamma_0)', \Delta', \text{SC})\).

**Proof.** \((\Gamma')(T) \leq \Gamma'(T)\), since \(\Gamma'(T)\) is a coperturbation function. The other inequality is clear, noting that \(\Gamma'(T) = \Gamma'(J_{Y} T) \leq \Gamma'(q_M J_{Y} T), M \in I_0(Y\sim)\) \([7, 2.1(II) and 2.3(II)]\) and \(\Gamma' \leq \Delta' \leq (\Gamma_0)' \leq \rho \leq \kappa, \Delta' \leq \text{SC}\). ■

In the rest of this section we will suppose that \(\Gamma\) is a coperturbation function. We will make use of the following lemma concerning quotient spaces.

**Lemma 3.5.** Let \(M\) be a closed subspace of \(X\). Then
(i) For any closed subspace \(Z\) of \(X\) such that \(Z \supseteq M, X/Z \cong (X/M)/(Z/M)\).
(ii) For any closed subspace \(N\) of \(X/M\) there exists a closed subspace \(Z\) of \(X\) such that \(Z \supseteq M\) with \(q_N = q_M q_M\).

**Proof.** See \([7, 8.6(I)]\). ■

**Theorem 3.6.** Let \(T, S \in L(X, Y)\) such that \((\Delta')(J_{Y} S) < (\Gamma')(T)\) and \(D(S) \supseteq D(T)\). Then \(T+S \in F(X, Y)\).

**Proof.** As an immediate consequence of Lemma 3.5 we get that if \(Z \in I_0(Y\sim)\) then \((\Gamma')(T) \leq (\Gamma')(q_M J_{Y} T)\).

Suppose \((T+S) \notin F(X, Y)\), then by \([4, 4.3]\) there is \(M \in I_0(Y\sim)\) for which \(q_M J_{Y} (T+S)\) is compact and consequently \((\Gamma')(q_M J_{Y} T) = (\Gamma')(q_M J_{Y} S)\). Hence, \((\Gamma')(T) \leq (\Gamma')(q_M J_{Y} T) = (\Gamma')(q_M J_{Y} S) \leq (\Delta')(J_{Y} S)\), a contradiction and then \(T+S \in F(X, Y)\). ■

This theorem is the analogue of \([7, 3.5(VI)]\).

**Theorem 3.7.** Let \(T \in L(X, Y)\). Then
(i) \(T \in F(X, Y)\) if and only if \((\Gamma')(T) > 0\).
(ii) \(T \in \text{SC}(X, Y)\) if and only if \((\Delta')(T) = 0\).

**Proof.** (i) If \((\Gamma')(T) > 0\), by taking \(S = 0\) in the above theorem we have \(T \in F(X, Y)\). Suppose \(T \in \text{SC}(X, Y)\), then \(J_{Y} T \in F(X, Y\sim)\) and there exists \(F \in F(Y\sim)\) for which \((q_{F} J_{Y} T)'\) has a continuous inverse, i.e. \(j(T'_{F\perp}) > 0\) where \(F\perp\) is the annihilator of \(F\) in \((Y\sim)'\).
Let $M \in \text{I}_d(Y)$, then $M+F \in \text{I}_d(Y)$ since \( \dim(M+F)/M < \infty \) and \( (Y^\sim/M)/(M+F)/M \geq Y^\sim/M+F \). Consequently,
\[
0 < \gamma((q_{M+F}J_YT)) = \gamma((q_{M+F}J_YT)) = (q_{M+F}J_YT) \leq (q_{M+F}J_YT) \leq (q_{M+F}J_YT).
\]
Thus, \( \gamma(T) > 0 \).

(ii) \( T \in \text{SC}(X,Y) \Rightarrow \gamma(T) = 0 \). This assertion follows trivially from the fact that \( \gamma(T) \leq \Delta(T) \) and \( T \) is strictly cosingular if and only if \( \Delta(T) = 0 \) [4, 4.15]. The opposite implication follows from (i) and the fact that \( T \) is strictly cosingular if and only if for each \( M \in \text{I}_d(Y) \), \( q_M T \notin \text{F}_-(X,Y) \) [9, 1.2].

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**References**


