SOME PROPERTIES IN SPACES OF MULTILINEAR FUNCTIONALS AND SPACES OF POLYNOMIALS

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Abstract

In this paper we study some properties related to the biduals of certain Banach spaces whose elements are multilinear functionals, or polynomials, defined in Banach spaces. In particular, we obtain a simple description of the $n$-fold Schwartz $\epsilon$-product. We give examples of Banach spaces that are not quasi-reflexive and so, for each positive integer $n$, $P^n(X)^{**}$ identifies with $P^n(X)$ in a way clearly specified.

In the last fifteen years the theory of polynomials and multilinear forms has been extensively developed. The relationship between reflexivity and weak continuity was found by Ryan [26], the connection between reflexivity and weak sequential continuity was obtained by Alencar et al. [2], the use of upper and lower estimates and spreading models in connection with reflexivity was obtained by Farmer [12] and developed by Farmer and Johnson [13], Gonzalo [16] and Gonzalo and Jaramillo [17], [18]. Furthermore, Ryan [26] introduced the square order system for tensors and polynomials and Alencar [1] (see also [11]) discovered the connection between reflexivity of spaces of polynomials and the existence of a basis. Alencar et al. [2] first discussed polynomials on Tsirelson’s space and its dual, while Aron and Dineen [5] introduced Q-reflexive spaces and the theory of polynomials on the James–Tsirelson space.

In this paper we discuss refinements of some of the above results. In particular we show that the approximation property is not required for some of these results and we also give an example of a Q-reflexive space that is not quasi-reflexive. We also give new proofs and presentations of some results from the papers quoted above.

Throughout this paper we use infinite dimensional linear spaces defined over the field of real or complex numbers. $\mathbb{N}$ is the set of positive integers. For a set $B$, $|B|$ denotes its cardinal number.

If $X$ is a Banach space, then $X^*$ and $X^{**}$ represent its conjugate and double conjugate, respectively. $B(X)$ is the closed unit ball of $X$ and $\| \cdot \|$ its norm. If $A$ is a closed and bounded absolutely convex subset of $X$, $X_A$ will denote the linear hull of $A$ normed by its Minkowski functional; $X_A$ is then a Banach space. If $x \in X$ and $u \in X^*$, $(x, u)$ means $u(x)$. If $(x_n)$ is a sequence in $X$, $[x_n]$ will be its closed linear span. We say that $(x_n)$ is a seminormalised sequence if there exist $0 < h < k < \infty$ such that $h \leq \| x_n \| \leq k$, $n = 1, 2, \ldots$. If the sequence $(x_n)$ is basic, then $(x_n^*)$ is the

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sequence in \([x_n]^*\) formed by the linear functionals associated with the Schauder basis \((x_n)\) of \([x_n]\). We say that \(X\) is quasi-reflexive if it has finite codimension in its bidual; in particular, every reflexive Banach space is quasi-reflexive. A Banach space \(X\) is said to be Asplund when every separable closed subspace of \(X\) has separable dual, or equivalently, \(X^*\) has the Radon–Nikodym property.

A sequence \((x_n)\) in a Banach space \(X\) has a lower \(p\)-estimate \((1 \leq p < \infty)\) if there is a constant \(c > 0\) such that

\[
\left\| \sum_{n=1}^{m} a_n x_n \right\| \geq c \left( \sum_{n=1}^{m} |a_n|^p \right)^{1/p}
\]

for every finite sequence of scalars \(a_1, a_2, \ldots, a_m\). If \((x_n)\) is a basic sequence, it follows that if

\[
x = \sum_{n=1}^{\infty} a_n x_n \in [x_n],
\]

then \((a_n)\) is in \(l_p\) and there is a one-to-one continuous linear map \(T\) from \([x_n]\) into \(l_p\) such that

\[
T x_n = e_n, \quad n = 1, 2, \ldots,
\]

where \((e_n)\) is the unit vector basis of \(l_p\). We say that the sequence \((x_n)\) has an upper \(p\)-estimate if there is a constant \(c > 0\) such that

\[
\left\| \sum_{n=1}^{m} a_n x_n \right\| \leq c \left( \sum_{n=1}^{m} |a_n|^p \right)^{1/p}
\]

for every finite sequence \(a_1, a_2, \ldots, a_m\). Similarly, if \((a_n)\) belongs to \(l_p\), then \(\sum_{n=1}^{\infty} a_n x_n\) converges in \(X\) and hence there is a continuous linear map \(T\) from \(l_p\) into \(X\) such that

\[
T e_n = x_n, \quad n = 1, 2, \ldots.
\]

When the sequence \((x_n)\) is basic the map \(T\) is then one-to-one.

If \(X_1, X_2, \ldots, X_n\) are Banach spaces, we take the product \(B(X_1) \times B(X_2) \times \cdots \times B(X_n)\) as the closed unit ball of \(X_1 \times X_2 \times \cdots \times X_n\). Now, if \(Y\) is another Banach space, we put \(\mathcal{L}(X_1, X_2, \ldots, X_n; Y)\) for the Banach space of all continuous \(n\)-linear mappings from \(X_1 \times X_2 \times \cdots \times X_n\) into \(Y\) with the usual norm. When \(Y\) is the Banach space of scalars, we write \(\mathcal{L}(X_1, X_2, \ldots, X_n)\). \(\mathcal{K}(X_1, X_2, \ldots, X_n; Y)\) is the subspace of \(\mathcal{L}(X_1, X_2, \ldots, X_n; Y)\) formed by the compact \(n\)-linear mappings. \(\mathcal{L}_w'(X_1^*, X_2^*, \ldots, X_n^*; Y)\) will be the subspace of \(\mathcal{L}(X_1^*, X_2^*, \ldots, X_n^*; Y)\) formed by those elements whose restrictions to \(B(X_1^*) \times B(X_2^*) \times \cdots \times B(X_n^*)\) are \(w^*\)-norm continuous; it is norm-closed in \(\mathcal{L}(X_1^*, X_2^*, \ldots, X_n^*; Y)\) and it is immediate from the definition that each \(f \in \mathcal{L}_w'(X_1^*, X_2^*, \ldots, X_n^*; Y)\) is a compact \(n\)-linear mapping, i.e. \(f \in \mathcal{K}(X_1^*, X_2^*, \ldots, X_n^*; Y)\). Again, if \(Y\) is the space of scalars, we write \(\mathcal{L}_w'(X_1^*, X_2^*, \ldots, X_n^*)\). By \(\mathcal{L}_w'(X_1^*, X_2^*, \ldots, X_n^*)\) we denote the closure of \(\mathcal{L}_w'(X_1^*, X_2^*, \ldots, X_n^*)\) in \(\mathcal{L}(X_1^*, X_2^*, \ldots, X_n^*)\) with respect to the topology of the uni-
form convergence over the compact subsets of $X_1 \times X_2 \times \cdots \times X_n$. We suppose that $L^\infty(X_1, X_2, \ldots, X_n)$ is endowed with the norm induced by that of $L(X_1, X_2, \ldots, X_n)$.

We represent by $L^\infty(X_1, X_2, \ldots, X_n; Y)$ the subspace of $L(X_1, X_2, \ldots, X_n; Y)$ whose elements, when restricted to $B(X_1) \times B(X_2) \times \cdots \times B(X_n)$, are weak-norm continuous. Again, if $Y$ is the space of scalars, we write $L^\infty(X_1, X_2, \ldots, X_n)$. Whenever $X = X_1 = X_2 = \cdots = X_n$, we shall write $L^\infty(X; Y)$, $L^\infty(X_1)$, $L^\infty(X_2)$, $L^\infty(X^*)$, and $L^\infty(X)$, respectively, for the corresponding spaces defined above.

We recall Schwartz’s definition of the $\epsilon$-product of Banach spaces: $X_1^*$ is $X^*$-equipped with the topology $\gamma$ of uniform convergence on all compact sets of the Banach space $X$; note that $\gamma = \sigma(X^*, X)$ on $B(X^*)$. Now the $\epsilon$-product is defined by

$$X\epsilon Y := L(X_1^*; Y) \subset L(X^*; Y) = L(X^*, Y^*)$$

equipped with the operator norm; (see [21, pp 343–50]) for more details, in particular for the associativity $(X_1\epsilon X_2)\epsilon X_3 = X_1\epsilon (X_2\epsilon X_3)$ which allows us to define the $n$-fold $\epsilon$-product $X_1\epsilon X_2\epsilon \ldots\epsilon X_n$ of the Banach spaces $X_j$, $j = 1, 2, \ldots, n$.

**Theorem 1.** Let $X_1, X_2, \ldots, X_n, Y$ be Banach spaces. Then

$$L^\epsilon(X_1^*, X_2^*, \ldots, X_n^*; Y) = L^\epsilon(X_1^*, X_2^*, \ldots, X_n^*; Y) = L^\epsilon((X_1\epsilon X_2\epsilon \ldots\epsilon X_n)^*; Y) = X_1\epsilon X_2\epsilon \ldots\epsilon X_n\epsilon Y,$$

in particular

$$L^\epsilon(X_1^*, X_2^*, \ldots, X_n^*) = X_1\epsilon X_2\epsilon \ldots\epsilon X_n.$$

Observe that the latter equality is an isometry since both spaces carry the norm from $L^\epsilon(X_1^*, X_2^*, \ldots, X_n^*)$.

**Proof.** By the theorem of Banach–Dieudonné [22, p. 272], a subset $D \subset X_1^* \times X_2^* \times \cdots \times X_n^*$ is closed if and only if for all $\lambda > 0$ the set $D \cap \lambda[B(X_1^*) \times B(X_2^*) \times \cdots \times B(X_n^*)]$ is weak*-closed. This implies

$$L^\epsilon(X_1^*, X_2^*, \ldots, X_n^*; Y) \subset L^\epsilon(X_1^*, X_2^*, \ldots, X_n^*; Y),$$

the converse inequality being obvious. In particular, if $X$ is a Banach space,

$$L^\epsilon(X^*; Y) = L(X^*; Y) = X\epsilon Y.$$

We have

$$(X_1\epsilon X_2\epsilon \ldots\epsilon X_n)^* = X_1^* \otimes \cdots \otimes X_n^*$$

(see [21, p. 346]), hence we obtain

$$L^\epsilon(X_1^*, X_2^*, \ldots, X_n^*; Y) = L(X_1^*, X_2^*, \ldots, X_n^*; Y) = L((X_1\epsilon X_2\epsilon \ldots\epsilon X_n)^*; Y) = X_1\epsilon X_2\epsilon \ldots\epsilon X_n\epsilon Y.$$
Using the associativity \((X_1 \epsilon X_2 \epsilon \ldots \epsilon X_r) e (X_{r+1} \epsilon X_{r+2} \epsilon \ldots \epsilon X_n)\), we have

**Corollary 1.**

\[
\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*) = \mathcal{L}_w(X_1^*, X_2^*, \ldots, X_r^*; \mathcal{L}_w(X_{r+1}^*, X_{r+2}^*, \ldots, X_n^*)),
\]

in particular

\[
\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*) = \mathcal{L}_w(X_1^*, X_2^*, \ldots, X_{n-1}^*; X_n^*).
\]

Again, these equalities are isometries. Note that every element in \(\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*)\) defines a compact operator

\[
X_1^* \otimes X_2^* \otimes \cdots \otimes X_{n-1}^* \rightarrow X_n^*.
\]

Given a Banach space \(X\) and a positive integer \(n\), if we consider the restrictions to the diagonal of the elements of \(\mathcal{L}(\mathcal{P}(X))\), we obtain a linear space that we represent as \(\mathcal{P}(\mathcal{P}(X))\) so that if \(P \in \mathcal{P}(\mathcal{P}(X))\) there is an element \(f\) of \(\mathcal{L}(\mathcal{P}(X))\) such that

\[
P(x) := f(x, x, \ldots, x), \quad x \in X.
\]

We then say that \(P\) is a continuous \(n\)-homogeneous polynomial defined in \(X\). We define

\[\|P\| := \operatorname{sup}\{|P(x)| : \|x\| \leq 1\}\]

and assume that \(\mathcal{P}(\mathcal{P}(X))\) is provided with this norm.

As for tensor products, we shall use the notation of [23], extended here in the case of tensor products of more than two factors.

Given the Banach space \(X\) and the positive integer \(n\), \(\mathcal{P}_w(\mathcal{P}(X))\) is the subspace of \(\mathcal{P}(\mathcal{P}(X))\), whose elements are weak-continuous when restricted to \(B(X)\). \(\mathcal{P}_w(\mathcal{P}(X))\) is the subspace of \(\mathcal{P}(\mathcal{P}(X))\) formed by those elements whose restrictions to \(B(X)\) are weak*-continuous. \(\mathcal{P}_w(\mathcal{P}(X))\) is the closure of \(\mathcal{P}_w(\mathcal{P}(X))\) in \(\mathcal{P}(\mathcal{P}(X))\) for the topology of the uniform convergence over the compact subsets of \(X^*\). We assume that \(\mathcal{P}_w(\mathcal{P}(X))\) is provided with the norm induced by that of \(\mathcal{P}(\mathcal{P}(X))\).

Later on we shall make use of the following results that we proved in [28]:

(a) If \(X_1, X_2, \ldots, X_n\) are Asplund spaces, then \(\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*)\) is also an Asplund space.

(b) Let \(X\) be a Banach space and let \(n\) be a positive integer. If \(\mathcal{P}_w(\mathcal{P}(X))\) does not contain copies of \(l_1\), or, in particular, if \(X\) is an Asplund space, then

\[\mathcal{P}_w(\mathcal{P}(X))^{**} = \mathcal{P}_w(\mathcal{P}(X)).\]

(c) Let \(X\) be a Banach space and \(n\) a positive integer. If \(\mathcal{P}_w(\mathcal{P}(X))\) has no copy of \(l_1\), or, in particular, if \(X\) is Asplund, then, if \(X^*\) has the approximation property, we have that

\[\mathcal{P}_w(\mathcal{P}(X))^{**} = \mathcal{P}(X^*).\]
The following result [19] will also be needed:

(d) Let $X$ be a Banach space with no copies of $l_1$. If $A$ is a weak$^*$-compact subset of $X^*$, then its norm closed absolutely convex hull is weak$^*$-compact.

**Theorem 2.** If $X_1, X_2, \ldots, X_n$ are Banach spaces such that $\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*)$ has no copy of $l_1$, then

$$\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*)^{**} = \mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*).$$

**Proof.** If we call

$$Y := \mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*),$$

let $\psi$ be the continuous linear map from $X := X_1^* \otimes_\pi X_2^* \otimes_\pi \cdots \otimes_\pi X_n^*$ into $Y^*$ such that, if $f$ is in $Y$ and $u_j$ is in $X_j^*$, $j = 1, 2, \ldots, n$, then

$$\langle f, \psi(u_1 \otimes u_2 \otimes \cdots \otimes u_n) \rangle = f(u_1, u_2, \ldots, u_n).$$

We set

$$M := \{u_1 \otimes u_2 \otimes \cdots \otimes u_n \in X : \|u_j\| \leq 1, j = 1, 2, \ldots, n\}.$$ 

It is immediate that $\psi(M)$ is weak$^*$-compact in $Y^*$ and that $B(Y)$ is the polar set of $\psi(M)$ in $Y$. By result d, the closed absolutely convex hull of $\psi(M)$ in $Y^*$ coincides with $B(Y^*)$. Besides, the closed absolutely convex hull of $M$ in $X$ equals $B(X)$. Thus, $\psi(B(X))$ is dense in $B(Y^*)$ and, after the closed graph theorem, we have that $\psi(B(X))$ contains the interior of $B(Y^*)$ in $Y^*$, [20, p. 296]. Consequently, $\psi$ is an onto map. If

$$\psi^*: Y^{**} \rightarrow X^*$$

is the conjugate map of $\psi$, then $\psi^*$ is one-to-one and $\psi^*(Y^{**})$ is $\sigma(X^*, X)$-closed [22, p. 18]. On the other hand, $\psi^*(B(Y^{**}))$ is $B(X^*) \cap \psi^*(Y^{**})$. Hence if we identify $X^*$ with $\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*)$ in the usual manner, it follows that, bearing in mind that in $B(Y^{**})$ the topology $\sigma(Y^{**}, Y^*)$ is the same as that of the uniform convergence over the compact subsets of $Y^*$, $\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*)$ coincides with $\psi^*(Y^{**})$ and the result follows.

**Corollary 2.** If $X_1, X_2, \ldots, X_n$ are Asplund spaces, then

$$\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*)^{**} = \mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*).$$

**Proof.** After result (a), $\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*)$ is an Asplund space and hence it contains no copy of $l_1$. Our former theorem now applies.

**Lemma 1.** Let $X_1, X_2, \ldots, X_n, Y$ be Banach spaces. Let $r$ be a non-negative integer less than $n$. If $f \in \mathcal{L}_w(X_1, X_2, \ldots, X_n; Y)$ and

$$f(x_1, x_2, \ldots, x_r, \ldots), \ (x_1, x_2, \ldots, x_r) \in \prod_{j=1}^r B(X_j),$$

then
is the family of \((n-r)\)-linear mappings defined in \(\prod_{j=r+1}^n X_j\) taking values in \(Y\), with \((x_1, x_2, \ldots, x_r)\) as parameter, we then have that the family formed by the restrictions of the elements of (1) to \(\prod_{j=r+1}^n B(X_j)\) is weakly equicontinuous at the origin.

**Proof.** We proceed by induction over \(n\). For \(n = 1\), it happens that \(r = 0\) and the family in (1) has only one element; the property then clearly holds. We assume now that the property is true for \(n = 1, 2, \ldots, m-1\). We fix \(n = m\) and proceed inductively over \(r\). If \(r = 0\), again (1) has only one function and the property follows. Let \(s\) be a positive integer less than \(m\) and assume that the property holds for \(r = s - 1\), but it does not hold for \(r = s\). Then there is \(\delta > 0\) such that, for each weak neighborhood \(U\) of the origin in \(B(X_{s+1}) \times B(X_{s+2}) \times \cdots \times B(X_m)\), we have that

\[
\sup\{\|f(x_1, x_2, \ldots, x_m)\| : x_j \in B(X_j), j = 1, 2, \ldots, s, x_{s+1}, x_{s+2}, \ldots, x_m) \in U\} > \delta. \tag{2}
\]

Now, since the property is true for \(r = s - 1\), there is a weak neighborhood \(U_j\) of the origin in \(B(X_{j})\), \(j = s, s + 1, \ldots, m\), so that, for any value of the parameter \((x_1, x_2, \ldots, x_{s-1})\),

\[
\|f(x_1, x_2, \ldots, x_s, x_{s+1}, \ldots, x_m)\| < \frac{1}{s} \delta, \quad x_j \in U_j, \quad j = s, s + 1, \ldots, m. \tag{3}
\]

We put \(U_{ij} := U_j, \ j = s + 1, s + 2, \ldots, m\), and find, bearing (2) in mind,

\[
x_{1j} \in B(X_j), \quad j = 1, 2, \ldots, s, \quad x_{1j} \in U_{ij}, \quad j = s + 1, s + 2, \ldots, m,
\]

such that

\[
\|f(x_{11}, x_{12}, \ldots, x_{1m})\| > \delta.
\]

Proceeding again by complete induction, let us suppose that, for a positive integer \(k\), we have obtained a weak neighborhood \(U_{kj}\) of the origin in \(B(X_j)\), \(j = s + 1, s + 2, \ldots, m\), and

\[
x_{kj} \in B(X_j), \quad j = 1, 2, \ldots, s, \quad x_{kj} \in U_{kj}, \quad j = s + 1, s + 2, \ldots, m,
\]

such that

\[
\|f(x_{k1}, x_{k2}, \ldots, x_{km})\| > \delta. \tag{4}
\]

We have now that \(f(\ldots, x_k, \ldots)\) is a \(Y\)-valued \((m-1)\)-linear mapping whose restriction to \(B(X_1) \times \cdots \times B(X_{s-1}) \times B(X_{s+1}) \times \cdots \times B(X_m)\) is weakly continuous and hence the restrictions to \(\prod_{j=s+1}^m B(X_j)\) of the \((m-s)\)-linear mappings

\[
f(x_1, x_2, \ldots, x_{s-1}, x_{s+1}, \ldots), \quad (x_1, x_2, \ldots, x_{s-1}) \in \prod_{j=1}^{s-1} B(X_j)
\]

form a family which is weakly equicontinuous at the origin. Consequently, there is a weak neighborhood \(U_{(k+1)j}\) of the origin in \(B(X_j)\), \(U_{(k+1)j} \subset U_{kj}, j = s + 1, s + 2, \ldots, m\), so that, for any value of the parameter \((x_1, x_2, \ldots, x_{s-1})\)

\[
\|f(x_1, x_2, \ldots, x_{s-1}, x_{s+1}, \ldots, x_m)\| < \frac{1}{s} \delta, \quad x_j \in U_{(k+1)j}, \quad j = s + 1, s + 2, \ldots, m. \tag{5}
\]
We apply (2) now and obtain
\[ x_{(k+1)}j \in B(X_j), \quad j = 1, 2, \ldots, s, \quad x_{(k+1)s} \in U_{(k+1)s}, \quad j = s + 1, s + 2, \ldots, m, \]
such that
\[ \|f(x_{(k+1)1}, x_{(k+1)2}, \ldots, x_{(k+1)m})\| > \delta, \]
which coincides with inequality (4) replacing \( k \) by \( k + 1 \). The sequence \((x_k)_{k=1}^{\infty}\) has a weak*-cluster point \( x_0 \) in \( B(X_+^*) \). We find a neighborhood \( V \) of the origin of \( X_+^* \) for the weak* topology, absolutely convex, such that \( V \cap B(X_s) \subset U_j \). We find two positive integers \( p < q \) such that \( x_{ps} \) and \( x_{qs} \) belong to \( x_0 + V \). We put, for every \((x_1, x_2, \ldots, x_{s-1}) \in \prod_{j=1}^{s-1} B(X_j), \)
\[ I(x_1, x_2, \ldots, x_{s-1}) := f(x_1, x_2, \ldots, x_{s-1}, \frac{1}{2}(x_{qs} - x_{ps}), x_{q(s+1)}, \ldots, x_{qm}), \]
\[ J(x_1, x_2, \ldots, x_{s-1}) := f(x_1, x_2, \ldots, x_{s-1}, x_{ps}, x_{q(s+1)}, \ldots, x_{qm}), \]
\[ H := f(x_{q1}, x_{q2}, \ldots, x_{qm}). \]
It follows that
\[ \frac{1}{4}(x_{qs} - x_{ps}) \in U_s, \quad x_{qj} \in U_{qj} \subset U_{(p+1)j} \subset U_j, \quad j = s + 1, s + 2, \ldots, m, \]
and therefore, applying (3), (5) and (4), respectively, we obtain that
\[ \|I(x_1, x_2, \ldots, x_{s-1})\| < \frac{1}{4} \delta, \quad \|J(x_1, x_2, \ldots, x_{s-1})\| < \frac{1}{4} \delta, \quad \|H\| > \delta. \]
Then
\[ \frac{1}{4} \delta > \|I(x_{q1}, x_{q2}, \ldots, x_{q(s-1)})\| = \|\frac{1}{2} H - \frac{1}{2} J(x_{q1}, x_{q2}, \ldots, x_{q(s-1)})\| > \frac{1}{2} \delta - \frac{1}{8} \delta > \frac{1}{4} \delta, \]
which is a contradiction. \( \blacksquare \)

**Theorem 3.** Let \( X_1, X_2, \ldots, X_n, Y \) be Banach spaces. Let \( r \) be a non-negative integer less than \( n \). If \( f \in \mathcal{P}_r(X_1, X_2, \ldots, X_n, Y) \) and
\[ f(x_1, x_2, \ldots, x_r, \ldots, \), \quad (x_1, x_2, \ldots, x_r) \in \prod_{j=1}^{r} B(X_j), \quad (6) \]
is the family of \((n - r)\)-linear mappings defined in \( \prod_{j=r+1}^{n} X_j \) with values in \( Y \), \((x_1, x_2, \ldots, x_r)\) as parameter, then we have that the family formed by the restrictions of the elements of (6) to \( \prod_{j=r+1}^{n} B(X_j) \) is weakly uniformly equicontinuous in \( \prod_{j=r+1}^{n} B(X_j) \).

**Proof.** For each integer \( j, r < j \leq n \), we may apply Lemma 1, conveniently reordering the Banach spaces \( X_{r+1}, X_{r+2}, \ldots, X_n \) if necessary. We obtain that the family of mappings with parameter
\[ (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in B(X_1) \times \cdots \times B(X_{j-1}) \times B(X_{j+1}) \times \cdots \times B(X_n) \]
formed by the restrictions of the linear functionals

\[ f(x_1, \ldots, x_{j-1}, x_j+1, \ldots, x_n) \]

to \( B(X_j) \) is weakly equicontinuous at the origin. Thus, given \( \varepsilon > 0 \), there is an absolutely convex weak neighborhood of the origin \( V_j \) in \( X_j \) such that

\[ \| f(x_1, \ldots, x_{j-1}, v_j, x_{j+1}, \ldots, x_n) \| < \frac{1}{2n} \varepsilon, \quad v_j \in V_j \cap B(X_j), \]

\[ x_i \in B(X_i), \quad i = 1, 2, \ldots, j - 1, j + 1, \ldots, n. \]

Then, given the points,

\[ (y_{r+1}, y_{r+2}, \ldots, y_n), (z_{r+1}, z_{r+2}, \ldots, z_n) \in \prod_{j=r+1}^n B(X_j), \quad y_j - z_j \in V_j, \quad j = r + 1, r + 2, \ldots, n, \]

it follows that, for every \( x_i \in B(X_i), i = 1, 2, \ldots, r \), noticing that \( \frac{1}{2}(y_j - z_j) \in V_j \cap B(X_j), j = r + 1, r + 2, \ldots, n, \)

\[ \| f(x_1, x_2, \ldots, x_r, y_{r+1}, y_{r+2}, \ldots, y_n) - f(x_1, x_2, \ldots, x_r, z_{r+1}, z_{r+2}, \ldots, z_n) \| = \]

\[ \| 2f(x_1, x_2, \ldots, x_r, y_{r+1}, y_{r+2}, \ldots, y_n) + 2f(x_1, x_2, \ldots, x_r, z_{r+1}, \frac{1}{2}(y_{r+2} - z_{r+2}), \ldots, z_n) + \cdots + 2f(x_1, x_2, \ldots, x_r, y_{r+1}, y_{r+2}, \ldots, \frac{1}{2}(y_n - z_n)) \| < \varepsilon. \]

**Corollary 3.** Let \( X_1, X_2, \ldots, X_n, Y \) be Banach spaces. If \( f \in L^w(X_1, X_2, \ldots, X_n; Y) \), then \( f \) is weakly uniformly continuous in \( B(X_1) \times B(X_2) \times \cdots \times B(X_n) \).

**Corollary 4** [6]. Let \( P \) be an \( n \)-homogeneous polynomial defined in a Banach space \( X \) with values in a Banach space \( Y \). If \( P \) is weakly continuous on \( B(X) \) then \( P \) is weakly uniformly continuous on \( B(X) \).

We recall that Corollary 3 can also be obtained by conveniently modifying the method described in [6] to \( n \)-linear mappings.

**Note 1.** Notice that the method used in the proofs of Lemma 1 and Theorem 3 can be applied as to achieve the following result:

Let \( E_1, E_2, \ldots, E_n, F \) be locally convex spaces. Let \( A_j \) be an absolutely convex precompact subset of \( E_j \), \( j = 1, 2, \ldots, n \). Let \( f \) be an \( n \)-linear mapping of \( \prod_{j=1}^n E_j \) into \( F \) whose restriction to \( \prod_{j=1}^n A_j \) is continuous. If \( r \) is a non-negative integer less than \( n \) and

\[ f(x_1, x_2, \ldots, x_r, \ldots), \quad (x_1, x_2, \ldots, x_r) \in \prod_{j=1}^r A_j \quad (7) \]

is the family of \((n - r)\)-linear mappings defined in \( \prod_{j=r+1}^n E_j \) with values in \( F \), with parameter \((x_1, x_2, \ldots, x_r)\), then we have that the family formed by the restrictions of the elements of \( (7) \) to \( \prod_{j=r+1}^n A_j \) is uniformly equicontinuous in \( \prod_{j=r+1}^n A_j \).
Let \( X_1, X_2, \ldots, X_n \) be Banach spaces. If \( f \) belongs to \( \mathcal{L}_w(X_1, X_2, \ldots, X_n) \), it may be extended to an element \( \tilde{f} \) of \( \mathcal{L}_w(X_1^{**}, X_2^{**}, \ldots, X_n^{**}) \), according to Corollary 3. The mapping that takes \( f \) into \( \tilde{f} \) is a linear isometry. By means of this mapping, we identify \( \mathcal{L}_w(X_1, X_2, \ldots, X_n) \) with \( \mathcal{L}_w(X_1^{**}, X_2^{**}, \ldots, X_n^{**}) \). Similarly, for a Banach space \( X \) and a positive integer \( n \), we may identify \( \mathcal{P}_w(X) \) with \( \mathcal{P}_w(X^{**}) \), according to Corollary 4. Then, using Theorem 2 and Corollary 2, or else result (b), respectively, we obtain the following results.

**Proposition 1.** If \( X_1, X_2, \ldots, X_n \) are Banach spaces such that \( \mathcal{L}_w(X_1, X_2, \ldots, X_n) \) contains no copy of \( l_1 \), or, in particular, when \( X_1^{*}, X_2^{*}, \ldots, X_n^{*} \) are Asplund, then

\[
\mathcal{L}_w(X_1, X_2, \ldots, X_n)^{**} = \mathcal{L}_w(X_1^{**}, X_2^{**}, \ldots, X_n^{**}).
\]

**Proposition 2.** If \( X \) is a Banach space and \( n \) is a positive integer such that \( \mathcal{P}_w(X) \) has no copy of \( l_1 \), or, in particular, when \( X^* \) is Asplund, then

\[
\mathcal{P}_w(X)^{**} = \mathcal{P}_w(X^{**}).
\]

The reflexivity of spaces of polynomials as well as for spaces of multilinear functionals has been studied by a number of authors, e.g. [1]–[4], [12], [26]. We obtain here a few new results concerning reflexivity.

**Proposition 3.** Let \( X_1, X_2, \ldots, X_n \) be Banach spaces. The following conditions are equivalent:

1. \( \mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*) \) is reflexive.
2. \( \mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*) \) is sequentially weakly continuous in \( X_1^* \times X_2^* \times \cdots \times X_n^* \).

**Proof.** We write \( X := \mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*), \ Y := \mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*) \). 1 \( \Rightarrow \) 2. \( X_1 \odot X_2 \odot \cdots \odot X_n \) identifies canonically with a closed subspace of \( X \), hence we deduce, keeping in mind that in this paper we only deal with non-trivial linear spaces, that \( X_j \) is reflexive, \( j = 1, 2, \ldots, n \). On the other hand, \( X \) coincides with its bidual \( Y \) and the conclusion follows. 2 \( \Rightarrow \) 1. Let \( \psi \) denote the canonical mapping from \( X_1^* \times X_2^* \times \cdots \times X_n^* \) into \( X^* \) such that

\[
\langle f, \psi(u_1, u_2, \ldots, u_n) \rangle = f(u_1, u_2, \ldots, u_n), \quad f \in Y, \quad u_j \in X_j^*, \quad j = 1, 2, \ldots, n.
\]

We set

\[
M := \psi(B(X_1^*) \times B(X_2^*) \times \cdots \times B(X_n^*)).
\]

Let \( (z_j) \) be a sequence in \( M \). We choose \( u_{jr} \) in \( B(X_j^*) \), \( j = 1, 2, \ldots, n \), such that

\[
\psi(u_{1r}, u_{2r}, \ldots, u_{nr}) = z_r, \quad r = 1, 2, \ldots
\]

We find a subsequence \( (u_{j_{rn}}) \) of \( (u_{jr}) \) that converges in \( X_j^* \) to \( u_j \) for the weak topology. Then if

\[
z := \psi(u_1, u_2, \ldots, u_n) \quad \text{and} \quad f \in Y,
\]
it follows that
\[
\lim_m (f, z_m) = \lim_m f(u_{1m}, u_{2m}, \ldots, u_{nm}) = f(u_1, u_2, \ldots, u_n) = (f, z),
\]
and thus \( M \) is \( \sigma(X^*, X) \)-compact. Now, since \( B(X^*) \) is the closed convex hull of \( M \) in \( X^* \), we apply Krein’s theorem [21, p. 235] to obtain that \( B(X^*) \) is weakly compact and hence \( X \) is reflexive.

**Proposition 4.** Let \( X \) be a Banach space and \( n \) a positive integer. The following conditions are equivalent:

1. \( L_w^n(X^*) \) is reflexive.
2. \( X \) is reflexive and, for each integer \( r, 1 \leq r \leq n \), every element of \( L_{(w^*)^r}(X^*) \) is sequentially weakly continuous in the origin.

**Proof.** 1 \( \Rightarrow \) 2. If \( 1 \leq r \leq n \), \( L_w^n(X^*) \) is isomorphic to a subspace of \( L^n_w(X^*) \) and is thereby reflexive. This conclusion follows from Proposition 3. 2 \( \Rightarrow \) 1. To apply Proposition 3, take \( f \in L_w^n(X^*) \). Using with \( I := \{1, 2, \ldots, n\} \) the formula
\[
f(x_n + x) = f(x_n) + \sum_{0 \neq j \in I} f_{P_j(x)}(P_{1,J}x_n)
\]
from [3] (see proof of theorem 3.1.; here \( f_I := f_{P_I(x)}(P_{1,J}x) \) is just the \((n-|J|)\)-linear mapping \( f \) with \( x(j) \) fixed in the \( j \)-th component for \( j \in J \) and observing that \( f_J \in L^n_{(w^*)^{|I|}}(X^*) \), one immediately obtains the sequential weak continuity of \( f \).

**Proposition 5.** Let \( X \) be a Banach space and \( n \) a positive integer. The following conditions are then equivalent:

1. \( P_{w^n}(X^*) \) is reflexive.
2. \( X \) is reflexive and each element of \( P_{w^n}(X^*) \) is weakly sequentially continuous in \( X^* \).

**Proof.** 1 \( \Rightarrow \) 2. Let \( L_{w^n}(X^*) \) be the subspace of \( L_{w^n}(X^*) \) formed by the symmetric elements. If \( P \) belongs to \( P_{w^n}(X^*) \), we write \( T(P) \) for the \( n \)-linear symmetric functional obtained from \( P \) by means of the polarisation formula. Then
\[
T \colon P_{w^n}(X^*) \longrightarrow L_{w^n}(X^*)
\]
is an isomorphism. Now, take \( u_0 \) in \( X^* \), \( u_0 \neq 0 \). For every \( g \) in \( L_{w^n}(X^*) \), let \( Sg \) be the element of \( X \) such that, for each \( u \) in \( X^* \),
\[
(Sg)(u) = g(u, u_0, u_0, \ldots, u_0).
\]
Then
\[
S \circ T \colon P_{w^n}(X^*) \longrightarrow X
\]
is a continuous onto linear map and hence \( X \) is reflexive. On the other hand, we
apply result (b) to obtain that $\mathcal{P}_w(\beta X^*)$ coincides with $\mathcal{P}_w(\beta X^*)$ and the result follows. 2 \Rightarrow 1. Let $\psi$ be the injection from $X^*$ into $\mathcal{P}_w(\beta X^*)$ such that

$$\langle P, \psi(u) \rangle = P(u), \quad P \in \mathcal{P}_w(\beta X^*), \quad u \in X^*.$$  

Proceeding now as in the proof of Proposition 3, we obtain that $\psi(B(X^*))$ is a $\sigma(\mathcal{P}_w(\beta X^*), \mathcal{P}_w(\beta X^*))$-compact and, by Krein’s theorem, it follows that $B(\mathcal{P}_w(\beta X^*))$ is weakly compact and thus $\mathcal{P}_w(\beta X^*)$ is reflexive. 

\textbf{Proposition 6.} Let $X$ be a Banach space and let $n$ be a positive integer. The following conditions are equivalent:

(1) $\mathcal{P}_w(\beta X^*)$ is reflexive.

(2) $X$ is reflexive and, for each integer $r$, $1 \leq r \leq n$, every element of $\mathcal{P}_w(\beta X^*)$ is sequentially weakly continuous in the origin of $X^*$.

\textbf{Proof.} 1 \Rightarrow 2. We consider $T$ and $u_0$ with the same meaning as in the proof of Proposition 5. We fix $1 \leq r \leq n$. For each $g$ in $\mathcal{L}^1_{\omega}(\beta X^*)$, let $Vg$ be the element of $\mathcal{P}_w(\beta X^*)$ such that, for each $u$ in $X^*$,

$$(Vg)(u) = g(u, \ldots, u, u_0, \ldots, u_0).$$

Then,

$$V \circ T \colon \mathcal{P}_w(\beta X^*) \longrightarrow \mathcal{P}_w(\beta X^*)$$

is linear, continuous and onto and therefore $\mathcal{P}_w(\beta X^*)$ is reflexive. We apply Proposition 5 and thus have that $X$ is reflexive and that every element of $\mathcal{P}_w(\beta X^*)$ is sequentially weakly continuous in the origin of $X^*$. 2 \Rightarrow 1. The proof is analogous to that of Proposition 4, keeping in mind the isomorphism between $\mathcal{P}_w(\beta X^*)$ and $\mathcal{L}^1_{\omega}(\beta X^*)$ and taking $f$ to be symmetric. 

\textbf{Proposition 7.} Let $X_1, X_2, \ldots, X_n$ be Banach spaces. The following conditions are equivalent:

(1) For each closed subspace $Y_j$ of $X_j$, $j = 1, 2, \ldots, n$, $\mathcal{L}(Y_1, Y_2, \ldots, Y_n)$ is reflexive.

(2) $X_j$ is reflexive and, for each closed subspace $Y_j$ of $X_j$, $j = 1, 2, \ldots, n$, the elements of $\mathcal{L}(Y_1, Y_2, \ldots, Y_n)$ are sequentially weakly continuous.

\textbf{Proof.} 1 \Rightarrow 2. Since $X_j^*$ is isomorphic to a subspace of $\mathcal{L}(X_1, X_2, \ldots, X_n)$, it is reflexive. Take now a closed subspace $Y_j$ of $X_j$. Let $(x_{jm})$ be a sequence in $Y_j$ weakly convergent to $x_j$ that does not converge in norm. We may find a subsequence $(y_{jm})$ of $(x_{jm})$ such that $(y_{jm} - x_j)$ is a basic sequence, $j = 1, 2, \ldots, n$. Let $Z_j$ be the linear span of $\{x_j, y_{j1}, y_{j2}, \ldots, y_{jm} \}$, and $Z_j$ follows that $Z_j$ has a Schauder basis $(z_{jm})$. Now, let $(v_{jm})$ be a bounded sequence of $Y_j^*$ such that the restrictions of $v_{jm}$ to $Z_j$, $m = 1, 2, \ldots$, are the functionals associated with the basis $(z_{jm})$. Then there is an order $<^*$ in the set

$$M := \{z_{1m} \otimes z_{2m} \otimes \cdots \otimes z_{nm} : m = 1, 2, \ldots \}$$

such that $(M, <^*)$ is a Schauder basis of $Z := Z_1 \otimes Z_2 \otimes \cdots \otimes Z_n$, [14]. Then the
linear hull of \( \{ u_{1m_1} \otimes u_{2m_2} \otimes \cdots \otimes u_{mn} : m_1, m_2, \ldots, m_n \in \mathbb{N} \} \) is dense in \( L(Z_1, Z_2, \ldots, Z_n) \).

Given the positive integers \( r_1, r_2, \ldots, r_n \), we have that

\[
\lim_m (y_{1m} \otimes y_{2m} \otimes \cdots \otimes y_{nm}) = \lim_m (y_{1m} y_{2m} \cdots u_{nm}) = (x_1 \otimes x_2 \otimes \cdots \otimes x_n, u_{r_1} \otimes u_{r_2} \otimes \cdots \otimes u_{rm})
\]

and hence, for each element \( g \) of \( L(Y_1, Y_2, \ldots, Y_n) \) we have that

\[
\lim_g (y_{1m}, y_{2m}, \ldots, y_{nm}) = g(x_1, x_2, \ldots, x_n),
\]

which concludes the proof. 2 \( \Rightarrow \) 1. We take a closed subspace \( Y_j \) of \( X_j \), \( j = 1, 2, \ldots, n \). We set \( U_j := Y_j^c \), \( j = 1, 2, \ldots, n \). Then \( L_{w^*}(U_1^c, U_2^c, \ldots, U_n^c) \) coincides with \( L(Y_1, Y_2, \ldots, Y_n) \) and, applying Proposition 3, this space happens to be reflexive.

Making use of Propositions 5 and 6, respectively, we obtain the following two propositions.

**Proposition 8.** Let \( X \) be a Banach space and let \( n \) be a positive integer. The following conditions are equivalent:

1. For every closed subspace \( Y \) of \( X \), \( P^n(Y) \) is reflexive.
2. \( X \) is reflexive and, for every closed subspace \( Y \) of \( X \), the elements of \( P^n(Y) \) are sequentially weakly continuous.

**Proposition 9.** Let \( X \) be a Banach space and let \( n \) be a positive integer. The following conditions are equivalent:

1. For every closed subspace \( Y \) of \( X \), \( P^n(Y) \) is reflexive.
2. \( X \) is reflexive and, for every closed subspace \( Y \) of \( X \) and every integer \( r \), \( 1 \leq r \leq n \), the elements of \( P(Y) \) are sequentially weakly continuous in the origin.

In the setting of reflexive Banach spaces with the approximation property, Ryan [26] characterised the reflexivity of \( P^n(X) \) by the weak continuity of each element of \( P^n(X) \) on the unit ball and Alencar et al. [2, proposition 4] showed that it is enough to check this at the origin. In Propositions 8 and 9 we have removed the approximation property hypothesis.

In order to prove the coming proposition, we shall need the following results that we obtained in [29]:

1. In a Banach space \( X \), let \( (x_m) \) be a Schauder basis which has a lower \( p \)-estimate. If \( n \) is an integer with \( n \geq p \) and we put, for each \( P \) of \( P^n(X) \),

\[
T(P) := (P(x_m)),
\]

then the mapping \( T : P^n(X) \rightarrow l_\infty \)

is linear, continuous and onto.

2. In a Banach space \( X \), let \( (x_m) \) be a sequence that converges weakly to the
origin. If $n$ is an even positive integer and $P$ is a continuous $n$-homogeneous polynomial defined in $X$ such that $(P(x_m))$ does not converge to zero, then there is a subsequence $(y_m)$ of $(x_m)$ with a lower $n$-estimate.

(g) In a Banach space $X$, let $(x_m)$ be a sequence weakly convergent to the origin. Let $P$ be a continuous $n$-homogeneous polynomial on $X$ for some positive integer $n$. If $(P(x_m))$ does not converge to zero, then there is a basic subsequence $(z_m)$ of $(x_m)$ and a constant $a > 0$ such that

$$\| \sum_{m \in B} z_m^* \| \leq a |B|^{\frac{n}{n-1}}$$

for each non-empty finite subset $B$ of $\mathbb{N}$.

We shall also make use of the following result to be found in [13]:

(h) Let $(x_m)$ be a semi-normalised sequence in the Banach space $X$ such that there are two constants $a > 0$ and $b > 0$ with

$$\| \sum_{m \in B} x_m \| \leq a |B|^b$$

for each non-empty finite subset $B$ of $\mathbb{N}$. If $p$ is a number such that $1 < p < \frac{1}{b}$, then there is a subsequence of $(x_m)$ with an upper $p$-estimate.

Gonzalo [16, Corollary II.4.5], improving Farmer’s previous result [12, theorem 1.3], obtained the following: Let $n$ be a positive integer and $X$ be a Banach space such that every normalised weakly null sequence has no lower $n$-estimate. Then for any $r < n$, each $P$ in $\mathcal{P}(rX)$ is weakly sequentially continuous. Thus, assuming $X$ is reflexive, for every closed subspace $Y$ of $X$ with the approximation property, $\mathcal{P}(rY)$ will be reflexive.

Proposition 10 improves, for $n$ even, the above result in two ways: it removes the approximation property from the hypothesis; and moreover, it gives the thesis for $r = n$.

**Proposition 10.** Let $X$ be a Banach space and let $n$ be an even positive integer. The following conditions are equivalent:

1. For each closed subspace $Y$ of $X$, $\mathcal{P}(rY)$, $r \leq n$ is reflexive.
2. $X$ is reflexive and every semi-normalised sequence that converges weakly to the origin in $X$ has no lower $n$-estimate.

**Proof.** 1 $\Rightarrow$ 2. From what has been said before, $X$ is reflexive. Let us assume that there is a semi-normalised sequence $(y_m)$ in $X$ weakly convergent to the origin and with a lower $n$-estimate. We may assume that it is a basic sequence. Let $Y$ be the closed linear span of $(y_m)$. We apply result (e) and obtain $P$ in $\mathcal{P}(rY)$ such that $P(y_m) = 1$, $m = 1, 2, \ldots$. Then, keeping Proposition 9 in mind, we achieve a contradiction. 2 $\Rightarrow$ 1. Let us assume that (1) is not satisfied. Again, Proposition 9 gives us an integer $r$, $1 \leq r \leq n$, a closed subspace $Y$ of $X$ and an element $P$ of $\mathcal{P}(Y)$ which is not sequentially weakly continuous in the origin. If $r = n$, it suffices to apply result (f) to reach the conclusion. If $r < n$, we obtain, after result (g), a
semi-normalised basic sequence \((y_m)\) of \(Y\) weakly convergent to the origin and a constant \(a > 0\) such that
\[
\| \sum_{m \in B} y_m^* \| \leq a |B|^{\frac{1}{n}}
\]
for each non-empty finite subset \(B\) of \(\mathbb{N}\). Since \(\frac{1}{r} < \frac{n-1}{n}\), we apply result (h) to obtain a subsequence \((y_m^*)\) of \((y_m^*)\) and a continuous linear map
\[
T : l^\infty_n \rightarrow [y_n]^*
\]
so that, if \((e_m)\) is the unit vector basis of \(l^\infty_n\), then \(Te_m = y_m^*, m = 1, 2, \ldots\). Now, if
\[
T^* : [y_n] \rightarrow l_n
\]
is the conjugate map of \(T\), then the restriction of \(T^*\) to \([y_m]\) is a continuous one-to-one linear map that takes \((y_m)\) into the unit basis of \(l_n\). It follows then that \((y_m)\) has a lower \(n\)-estimate.

**Corollary 5.** If \(X\) is a Banach space, the following conditions are equivalent:

1. For each closed subspace \(Y\) of \(X\) and each positive integer \(n\), \(\mathcal{P}^n(Y)\) is reflexive.
2. \(X\) is reflexive and every semi-normalised sequence in \(X\) that weakly converges to the origin has no lower \(n\)-estimate for any positive integer \(n\).

The implication 2 \(\Rightarrow\) 1 of Corollary 5 was obtained by Farmer [12, theorem 1.3 (i)], for closed subspaces \(Y\) with the approximation property.

For the proof of the next proposition we need the following result, [29] (see also [18, proposition 1.9]):

(i) In a Banach space \(X\), let \((x_m)\) be a semi-normalised unconditional basic sequence weakly convergent to the origin. If there is, for a positive integer \(n\), a continuous \(n\)-homogeneous polynomial \(P\) defined in \(X\) such that \((P(x_m))\) does not converge to zero, then there is a subsequence \((y_m)\) of \((x_m)\) with a lower \(n\)-estimate.

**Proposition 11.** Let \(X\) be a Banach space such that every semi-normalised sequence in \(X\) that weakly converges to the origin has an unconditional basic subsequence. Let \(n\) be a positive integer. The following conditions are equivalent:

1. For each closed subspace \(Y\) of \(X\), \(\mathcal{P}^n(Y)\) is reflexive.
2. \(X\) is reflexive and every semi-normalised sequence in \(X\) that converges weakly to the origin has no lower \(n\)-estimate.

**Proof.** It suffices to follow an analogous argument to the one used in the proof of Proposition 10, making use of result (i) instead of result (f).

We do not know whether Proposition 10 holds for odd \(n\). Nevertheless, using result (g), one can show the following.

**Proposition 12.** Let \(X\) be a Banach space and let \(n\) be a positive integer. If there is a certain \(p > n\) such that every semi-normalised sequence in \(X\) that converges weakly to
the origin has no lower p-estimate, then \( \mathcal{P}(Y) \) is reflexive for every closed subspace \( Y \) of \( X \).

Proposition 12 removes the approximation property hypothesis in [16, corollary II.4.5] and [12, theorem 1.3] (see the remark above Proposition 10).

Next we study some properties of the space \( \mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*) \) using the approximation property in Banach spaces. The following theorem extends a well-known characterisation of the approximation property via compact linear operators or the \( e \)-product and a property of spaces of polynomials which can be found in [6].

**Theorem 4.** Let \( X_1, X_2, \ldots, X_{n-1} \) be Banach spaces. The following conditions are equivalent:

1. \( X_1, X_2, \ldots, X_{n-1} \) have the approximation property.
2. For each Banach space \( X_n \), \( X_1 \otimes X_2 \otimes \cdots \otimes X_n \) is dense in \( \mathcal{L}(X_1^*, X_2^*, \ldots, X_n^*) \).
3. For each reflexive Banach space \( X_n \), \( X_1 \otimes X_2 \otimes \cdots \otimes X_n \) is dense in \( \mathcal{L}(X_1^*, X_2^*, \ldots, X_n^*) \).
4. For each Banach space \( Y \), \( (X_1 \otimes X_2 \otimes \cdots \otimes X_{n-1}) \otimes Y \) is dense in \( \mathcal{L}(X_1^*, X_2^*, \ldots, X_n^*) \).

**Proof.**

1. \( \Rightarrow \) 2. Since \( X \otimes Y \) is dense in \( X \otimes Y \), if the Banach space \( X \) or the Banach space \( Y \) has the approximation property, it follows that \( X_{n-1} \otimes X_n \) is dense in \( X_{n-1} \otimes X_n \) and so \( X_{n-2} \otimes (X_{n-1} \otimes X_n) \) is dense in \( X_{n-2} \otimes (X_{n-1} \otimes X_n) \) and therefore \( X_{n-2} \otimes X_{n-1} \otimes X_n \) is dense in \( X_{n-2} \otimes X_{n-1} \otimes X_n \) and hence we obtain that \( X_1 \otimes X_2 \otimes \cdots \otimes X_n \) is dense in \( X_1 \otimes X_2 \otimes \cdots \otimes X_n \). 2. \( \Rightarrow \) 3. Obvious. 3. \( \Rightarrow \) 4. Using [9] (and [10, p. 33]), it is easy to see that for a compact set \( A \subset Y \) there is an absolutely convex compact set \( B \subset Y \) such that \( Y_B \) is reflexive and \( A \) is also compact in \( Y_B \). Since all \( f \in \mathcal{L}(X_1^*, X_2^*, \ldots, X_{n-1}; Y) \) are compact, this implies that one may assume \( Y \) to be reflexive in 4. Now

\[
\mathcal{L}(X_1^*, X_2^*, \ldots, X_{n-1}; Y) = \mathcal{L}(X_1^*, X_2^*, \ldots, X_{n-1}, Y^*)
\]

by Corollary 1, which gives 3 \( \Rightarrow \) 4 and 4 \( \Rightarrow \) 2. 2. \( \Rightarrow \) 1. To show that \( X := X_1 \epsilon \ldots \epsilon X_{n-1} \) has the approximation property (and hence \( X_1, \ldots, X_{n-1} \)) it is enough to show (see [10, p. 60]) that, for each reflexive Banach space \( X_n \), the space \( X \otimes X_n \) is dense in \( \mathcal{L}(X_n^*, X) = X_n \otimes X = \mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*) \). This is true by hypothesis. \( \blacksquare \)

**Note 2.** Schwartz [27] defines the \( e \)-product of \( n \) locally convex spaces \( E_i \) to be the set of \( n \)-linear mappings

\[
\phi : (E_1)_1 \times (E_2)_2 \times \cdots \times (E_n)_n \to K
\]

that are hypocontinuous with respect to the equicontinuous sets (i.e. if \( A_i \subset E_i \) are equicontinuous for \( i \neq i_0 \) then there is a neighborhood \( V_{i_0} \) of the origin in \( (E_{i_0})' \) with

\[
\sup_{i \neq i_0} |\phi(A_i \times V_{i_0})| \leq 1
\]
and proves that $E_1 \otimes E_2 \otimes \cdots \otimes E_n$ is dense if $E_1, E_2, \ldots, E_{n-1}$ have the approximation property.

**Corollary 6.** Let $X_1, X_2, \ldots, X_n$ be Banach spaces. If $X_1, X_2, \ldots, X_{n-1}$ have the approximation property, then

$$\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*) = X_1 \hat{\otimes} X_2 \hat{\otimes} \cdots \hat{\otimes} e X_n.$$  

As in [14], we endow $\mathbb{N}^2$ with the rectangular order $<^r$. Proceeding by induction, assume that for an integer $n - 1 \geq 2$ we have the elements of $\mathbb{N}^{n-1}$ ordered in the sequence $(a_i)$. We then order, in the rectangle manner, the pairs $(a_i, s)$, $a_i \in \mathbb{N}^{n-1}$, $s \in \mathbb{N}$, and thus define an order $<^r$ in $\mathbb{N}^n$. This is the square ordering described inductively by Ryan [26]. This ordering has also been used in [11].

**Corollary 7.** If $(x_{jm})$ is a Schauder basis in the Banach space $X_j$, $j = 1, 2, \ldots, n$, then

$$\{x_{1m_1} \otimes x_{2m_2} \otimes \cdots \otimes x_{nm_n}, (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n, <^r\}$$

is a Schauder basis in $\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*)$.

**Proof.** It follows immediately, applying [14] and Corollary 6. ■

**Corollary 8.** If $(x_{jm})$ is a shrinking basis in the Banach space $X_j$, $j = 1, 2, \ldots, n$, then

$$\{x_{1m_1}^* \otimes x_{2m_2}^* \otimes \cdots \otimes x_{nm_n}^*, (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n, <^r\}$$

is a Schauder basis in $\mathcal{L}_w(X_1, X_2, \ldots, X_n)$.

**Proposition 13.** Let $X_1, X_2, \ldots, X_n$ be Banach spaces such that $\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*)$ contains no copy of $1$. If $X_1^*, X_2^*, \ldots, X_{n-1}^*$ have the approximation property, then

$$\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*)^* = \mathcal{L}(X_1^*, X_2^*, \ldots, X_n^*).$$

**Proof.** Given $u_j \in X_j^*$, $j = 1, 2, \ldots, n$, the element $u_1 \otimes u_2 \otimes \cdots \otimes u_n$ of $\mathcal{L}(X_1^*, X_2^*, \ldots, X_n^*)$ clearly belongs to $\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*)$ and hence it suffices, after Theorem 2, to see that $X_1^* \otimes X_2^* \otimes \cdots \otimes X_n^*$ is dense in $X := \mathcal{L}(X_1^*, X_2^*, \ldots, X_n^*)$ for the topology of uniform convergence over each compact subset of $X_1^* \times X_2^* \times \cdots \times X_n^*$. The property is obviously true for $n = 1$. Following an induction argument, we assume that $X_2^* \otimes X_3^* \otimes \cdots \otimes X_n^*$ is dense in $Y := \mathcal{L}(X_2^*, X_3^*, \ldots, X_n^*)$ for the topology of the uniform convergence over the compact subsets of $X_2^* \times X_3^* \times \cdots \times X_n^*$. We take a compact subset $K_j$ contained in $B(X_j^*)$, $j = 1, 2, \ldots, n$. Let $f$ be an arbitrary element of $X$. Let $\epsilon > 0$. Let $T$ be the map from $X_1^*$ into $Y$ such that

$$Tu = f(u, \ldots), \quad u \in X_1^*.$$  

Then $T$ is continuous and linear. Since $X_1^*$ has the approximation property, there is
a finite-rank operator $S : X_1^* \rightarrow Y$ such that
\[ \| Tu_1 - Su_1 \| < \frac{1}{2} \epsilon, \quad u_1 \in K_1. \]
We may write
\[ S = \sum_{s=1}^{r} v_{1s} \otimes f_s, \quad v_{1s} \in X_1^{**}, \quad f_s \in Y, \quad s = 1, 2, \ldots, r. \]
We take \( \infty > k > \sum_{s=1}^{r} \| v_{1s} \| \) and find \( v_{hs}^{(s)} \in X_t^{**}, \quad h = 1, 2, \ldots, h(s), \quad i = 2, 3, \ldots, n, \) so that
\[ \left| f_s(u_2, u_3, \ldots, u_n) - \sum_{h=1}^{h(s)} v_{2h}^{(s)}(u_2)v_{3h}^{(s)}(u_3) \ldots v_{nh}^{(s)}(u_n) \right| < \frac{\epsilon}{2r} \]
\( u_i \in K_i, \quad i = 2, 3, \ldots, n, \quad s = 1, 2, \ldots, r. \)
Then, for \( u_i \in K_i, \quad i = 2, 3, \ldots, n, \)
\[ \left| f(u_1, u_2, \ldots, u_n) - \sum_{s=1}^{r} \sum_{h=1}^{h(s)} v_{1s}(u_1)v_{2h}^{(s)}(u_2) \ldots v_{nh}^{(s)}(u_n) \right| \]
\[ \leq \left| f(u_1, u_2, \ldots, u_n) - \sum_{s=1}^{r} v_{1s}(u_1)f_s(u_2, u_3, \ldots, u_n) \right| \]
\[ + \sum_{s=1}^{r} v_{1s}(u_1)f_s(u_2, u_3, \ldots, u_n) - \sum_{h=1}^{h(s)} v_{2h}^{(s)}(u_2)v_{3h}^{(s)}(u_3) \ldots v_{nh}^{(s)}(u_n) \]
\[ \leq \| Tu_1 - Su_1 \| + k \sum_{s=1}^{r} \frac{\epsilon}{2r} < \epsilon. \]

**Corollary 9.** Let \( X_1, X_2, \ldots, X_n \) be Banach spaces such that \( \mathcal{L}_w(X_1, X_2, \ldots, X_n) \) has no copies of \( l_1. \) If \( X_1^{**}, X_2^{**}, \ldots, X_{n-1}^{**} \) have the approximation property, then
\[ \mathcal{L}_w(X_1, X_2, \ldots, X_n)^{**} = \mathcal{L}(X_1^{**}, X_2^{**}, \ldots, X_{n-1}^{**}). \]

**Corollary 10.** Let \( X_j \) be a Banach space such that \( X_j^* \) is Asplund, \( j = 1, 2, \ldots, n. \) If \( X_1^{**}, X_2^{**}, \ldots, X_{n-1}^{**} \) have the approximation property, then
\[ \mathcal{L}_w(X_1, X_2, \ldots, X_n)^{**} = \mathcal{L}(X_1^{**}, X_2^{**}, \ldots, X_{n-1}^{**}). \]

**Proposition 14.** Let \( X \) be a Banach space and let \( n \) be a positive integer. If \( \mathcal{P}_n(X) \) contains no copy of \( l_1 \) or, in particular, if \( X^* \) is an Asplund space, then, if \( X^{**} \) has the approximation property, we have that
\[ \mathcal{P}_n(X)^{**} = \mathcal{P}(X^{**}). \]

**Proof.** An immediate consequence of result (c).
Lemma 2.

We start out with the following lemma.

The only known examples of Q-reflexive Banach spaces are either reflexive, as the dual of Tsirelson’s space, [24, p. 95], or quasi-reflexive [5]. In the following, we give examples of Q-reflexive Banach spaces that are neither reflexive nor quasi-reflexive. We start out with the following lemma.

**Lemma 2.**

\[
\mathcal{L}_w(X_1, X_2, \ldots, X_n) \subset \mathcal{L}_w(X_1 \bar{\otimes}_x X_2, X_3, \ldots, X_n).
\]

**Proof.** Each \( f \in \mathcal{L}_w(X_1, X_2, \ldots, X_n) \) defines a bilinear mapping

\[
T \in \mathcal{L}(X_1, X_2; \mathcal{L}_w(X_3, \ldots, X_n)).
\]

Let us prove that \( T \) is compact. For this take

\[
\tilde{f} \in \mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*),
\]

the canonical extension of \( f \) to \( X_1^* \times X_2^* \times \cdots \times X_n^* \). By Corollary 1 we have that

\[
\mathcal{L}_w(X_1^*, X_2^*, \ldots, X_n^*) = \mathcal{L}_w(X_1^*, X_2^*; \mathcal{L}_w(X_3^*, \ldots, X_n^*)).
\]

Let \( \tilde{T} \) be the element of \( \mathcal{L}_w(X_1^*, X_2^*; \mathcal{L}_w(X_3^*, \ldots, X_n^*)) \) associated with \( f \). Then \( \tilde{T} \) is compact and since \( T \) is the restriction of \( \tilde{T} \) to \( X_1 \times X_2 \), we obtain that \( T \) is compact also.

Let us denote by \( f_1 \in \mathcal{L}(X_1 \bar{\otimes}_x X_2, X_3, \ldots, X_n) \) the mapping related to \( f \) and by \( T_1 \in \mathcal{L}(X_1 \bar{\otimes}_x X_2; \mathcal{L}_w(X_3, \ldots, X_n)) \) the mapping related to \( T \). It is clear that \( T_1 \) is compact and that \( f_1(z, w) = T_1(z)[w], \ z \in X_1 \bar{\otimes}_x X_2, \ w \in X_3 \times \cdots \times X_n \). For \( S \in \mathcal{L}(X_3, \ldots, X_n) \) and \( w = (w_3, \ldots, w_n) \in X_3 \times \cdots \times X_n \) we shall use the notations \( S^{\oplus} \) for the elements of \( (X_3 \bar{\otimes}_x \cdots \bar{\otimes}_x X_n)^* \) associated with \( S \) and \( w^{\oplus} := w_3 \bar{\otimes}_x \cdots \bar{\otimes}_x w_n \).

Now take \( z_0 \to z \) in \( B(X_1 \bar{\otimes}_x X_2) \) weakly, \( w_0 \to w \) in \( B(X_1)^* \times \cdots \times B(X_n)^* \) and \( \varepsilon > 0 \). Since \( T_1 \) is compact, there are \( S_1, S_2, \ldots, S_m \in \mathcal{L}_w(X_3, \ldots, X_n) \) such that for every \( z \in B(X_1 \bar{\otimes}_x X_2) \) there is an integer \( r(z), \ 1 \leq r(z) \leq m \), with

\[
\| T_1(z) - S_{r(z)}^{\oplus} \| = \| T_1(z) - S_{r(z)} \| \leq \varepsilon.
\]

It is clear that \( S_{r(z)}^{\oplus} w_0^{\oplus} = S_{r(z)} w_0 \to S_{r(z)} w_0 = S_{r(z)} w_0^{\oplus} \). Therefore

\[
|f_1(z_0, w_0) - f_1(z_0, w_0)| \leq \| T_1(z_0)[w_0] - T_1(z_0)[w_0] \| + \| (T_1(z_0)^*, w_0^{\oplus} - w_0^{\oplus}) \|.
\]

The first term converges to zero since \( T_1(\cdot)[w_0] \in (X_1 \bar{\otimes}_x X_2)^* \) and the second term can be estimated by

\[
\max_{z} (2 \| T_1(z_0)^* - S_{r(z_0)}^{\oplus} \| + \| S_{r(z_0)}^{\oplus}, w_0^{\oplus} - w_0^{\oplus})
\]

which is smaller than \( 3\varepsilon \) for \( z \) sufficiently large. Thus \( f_1 \in \mathcal{L}_w(X_1 \bar{\otimes}_x X_2, X_3, \ldots, X_n) \).
Let $T^*$ be the dual of the Tsirelson space $T$ as it appeared in [24, p. 95] and $T_J^*$ the Tsirelson’–James’ space from [7] and [25].

**Theorem 5.** Let $Y$ be a subspace of $T^*$ with the approximation property. If

$$X := Y \widehat{\otimes}_\pi T_J^*,$$

then the following properties are satisfied:

1. $X$ has a topological complement in $X^{**}$ which is isomorphic to $Y$.
2. $X$ is $Q$-reflexive.

**Proof.** By [4], the spaces $Y$ and $T_J^*$ have the property $P_z$ for all $z > 0$ which implies (see [3]) that all $2n$-linear continuous mappings, $Y^n \times (T_J^*)^n \to \mathbb{K}$, are weakly sequentially continuous. Since $l_1 \not\in Y^n \times (T_J^*)^n$, they are in $\mathcal{L}_w(Y, T_J^*, \ldots, Y, T_J^*)$. In particular, by Lemma 2,

$$\mathcal{L}^w(nX) = \mathcal{L}_w(nX)$$

and hence Proposition 14 implies

$$\mathcal{P}(nX)^{**} = \mathcal{P}_w(nX)^{**} = \mathcal{P}(nX^{**}),$$

i.e. $X$ is $Q$-reflexive.

Moreover, using Corollary 4,

$$X^* = \mathcal{L}_w(Y, T_J^*) = \mathcal{L}_w(Y^{**}, T_J^{***}) = Y^* \widehat{\otimes}_\pi T_J^{**}$$

and since $T_J^{***} = T_J^* \oplus \mathbb{K}$,

$$X^{**} = Y^{**} \widehat{\otimes}_\pi T_J^{***} = Y \widehat{\otimes}_\pi (T_J^* \oplus \mathbb{K}) = (Y \widehat{\otimes}_\pi T_J^*) \oplus Y = X \oplus Y.$$  

If we consider a reflexive Banach space $Y$, it can be shown analogously to what we have done in Theorem 5, that $X := Y \widehat{\otimes}_\pi T_J^*$ has a topological complement in $X^{**}$ isomorphic to $Y$. Moreover, if $Y = l_p$ ($1 < p < \infty$) and $n$ is a positive integer with $n < p$, we have that

$$\mathcal{P}(nX)^{**} = \mathcal{P}(nX^{**}).$$

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*Added in proof.* After submitting our paper we got a preprint by González [15] where he obtained examples of $Q$-reflexive spaces that are neither reflexive nor quasi-reflexive.
References


