Abstract

We study norms and quasi-norms having large groups of isometries (uniquely maximal and almost transitive). It is shown that a uniquely maximal norm on a Banach space is its ‘best’ equivalent norm with respect to several concepts, such as uniform convexity and smoothness. A similar result for quasi-norms (on non-locally convex) spaces is given. We analyse the connection between uniquely maximal and almost transitive norms: it is proved that both properties coincide for super-reflexive spaces. The existence of an equivalent (quasi-) norm with many symmetries on a given (quasi-) Banach space has a considerable impact on the underlying linear topological structure: for example, an almost transitive Banach space having the Radon–Nikodym property must be super reflexive. A uniquely maximal quasi-normed space having a non-zero bounded linear functional is necessarily a normed space. Also, a uniquely maximal quasi-Banach space has exact type (at least when it is not super reflexive). This gives examples of Banach spaces that do not admit an almost transitive renorming and quasi-Banach spaces (including some Orlicz function spaces) that are not even quotients of any uniquely maximal quasi-Banach space.

0. Introduction

We deal in this paper with norms having a large group of symmetries. There are various ways of measuring the ‘size’ of the group of symmetries of a given norm on a Banach space [16], [15]. For example, the norm is called maximal if no equivalent norm gives a larger group of isometries. In general we cannot say that a maximal norm is the most symmetric norm because other equivalent norms with the same group of isometries could exist. We are concerned with norms that are determined by their groups of symmetries in the following sense.

Definition 0.1. A norm is said to be uniquely maximal if (it is maximal and) there is no equivalent norm with the same group of isometries, apart from its scalar multiples.

Obviously, one can replace ‘equivalent’ by ‘weaker’ in the definition. Cowie [4] characterised uniquely maximal norms by means of the action of the group of isometries on the unit sphere; he proved that a norm is uniquely maximal if and only if the convex hull of the orbits under the action of the group of isometries on the unit sphere is dense in the unit ball (that is to say, it is convex transitive). We are also interested in other stronger properties.