WEAK HOLOMORPHIC CONVERGENCE AND BOUNDING SETS IN BANACH SPACES

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Abstract
We show that weak holomorphic convergence and norm convergence coincide for sequences in a Banach space $E$ if and only if every bounding subset of $E$ is relatively compact.

Introduction

Let $\mathcal{H}(E)$ denote the vector space of all complex-valued holomorphic functions on a complex Banach space $E$. Following Petunin and Savkin [18] we say that a sequence $(x_n)$ in $E$ is $\mathcal{H}(E)$-convergent to a vector $x$ in $E$ if $f(x_n) \to f(x)$ for every $f$ in $\mathcal{H}(E)$. Petunin and Savkin [18] proved that $\mathcal{H}(E)$-convergence and norm-convergence coincide for sequences in $E$ if $E$ is weakly compactly generated, in particular if $E$ is separable or reflexive. On the other hand Aron, Choi and Llavona [3] showed that in $\ell^\infty$ these two convergence notions do not coincide. More precisely they showed that the sequence $(e_n)$ of unit vectors $\mathcal{H}(\ell^\infty)$ converges to zero, but does not converge to zero in norm.

Following Alexander [1] we say that a set $A$ in $E$ is bounding if every $f$ in $\mathcal{H}(E)$ is bounded on $A$. Hirschowitz [9] and Dineen [5] independently proved that every bounding set in $E$ is relatively compact if $E$ is separable and reflexive, and Schottenloher [21] remarked that the same conclusion holds if $E$ is weakly compactly generated (see also [6, p. 178]). On the other hand Dineen [4] proved that the sequence $(e_n)$ of unit vectors in $\ell^\infty$ is a bounding set that is not relatively compact (see also [6, theorem 4.31]).

Comparison of the aforementioned results suggests a close connection between the notions of $\mathcal{H}(E)$-convergence and bounding set. In this note we show that $\mathcal{H}(E)$-convergence and norm-convergence coincide for sequences in $E$ if and only if
every bounding subset of $E$ is relatively compact. We also establish a similar result for the notion of limited set.

Let us recall that a set $A \subseteq E$ is said to be limited if $\sup_{x \in A} |\varphi_n(x)| \to 0$ for every $\sigma(E',E)$-null sequence $(\varphi_n)$ in $E'$. Limited sets were studied by Mazur [15], Gelfand [8] and Phillips [19] in the thirties, and were reinvented 40 years later by Josefson [11] under the name of weakly bounding sets. Bounding sets are limited, and Banach spaces whose limited sets are relatively compact are called Gelfand–Phillips spaces. It is known that a Banach space is a Gelfand–Phillips space if the closed unit ball of its dual is weak-star sequentially compact (see [14]), and in particular if the space is weakly compactly generated (see [2]). Nevertheless the Banach space constructed by Josefson in [12] is a non Gelfand–Phillips space whose bounding sets are relatively compact. See also Schlumprecht [20] for a related example.

Our terminology is standard. For background information on infinite dimensional complex analysis we refer to the books of Dineen [6] or Mujica [16].

1. Weak holomorphic convergence and bounding sets

**Theorem 1.1.** For a Banach space $E$ the following conditions are equivalent:

(a) every bounding subset of $E$ is relatively compact;

(b) a sequence $(x_n)$ converges in $E$ if and only if $(f(x_n))$ converges in $\mathbb{C}$ for every $f \in \mathcal{H}(E)$.

**Proof.** Let $\mathcal{S}$ denote the family of all strictly increasing sequences in $\mathbb{N}$. We remark that $(x_n)$ is a Cauchy sequence in a topological vector space if and only if $x_{m_k} - x_{n_k} \to 0$ for each pair $(m_k), (n_k)$ in $\mathcal{S}$.

(a) $\implies$ (b): Suppose $(f(x_n))$ converges for every $f \in \mathcal{H}(E)$. Then the sequence $(x_n)$ is bounding, and therefore lies in a compact set $K$, by (a). Since $(f(x_n))$ is a Cauchy sequence, it follows that $f(x_{m_k}) - f(x_{n_k}) \to 0$ for each pair $(m_k), (n_k)$ in $\mathcal{S}$. In particular $x_{m_k} - x_{n_k} \to 0$ for the weak topology $\sigma(E', E)$. Since $K$ is norm compact, the norm topology and the weak topology coincide on $K$, and therefore $\|x_{m_k} - x_{n_k}\| \to 0$ for each pair $(m_k), (n_k)$ in $\mathcal{S}$. Thus $(x_n)$ is a Cauchy sequence in $E$, and therefore converges.

(b) $\implies$ (a): Let $A$ be a bounding subset of $E$, and let $(x_n)$ be a sequence in $A$. Since every bounding set is limited, a result in [3] guarantees that $(x_n)$ has a weakly Cauchy subsequence $(x_{n_k})$. Thus $(x_{n_k})$ is a limited, weakly Cauchy sequence and, by [7, corollary 2(iii)], $(P(x_{n_k}))$ converges for every continuous polynomial $P : E \to \mathbb{C}$.

Now let $f \in \mathcal{H}(E)$. Since the balanced hull of $2A$ is bounding (see [6, corollary 4.20]), it follows from the Cauchy integral formula (see [16, corollary 7.3]) that the Taylor series of $f$ at the origin converges to $f$ uniformly on $A$. Hence given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|f(x) - \sum_{m=0}^{N} P^m f(0)(x)| \leq \varepsilon$$

for every $x \in A$. Since the sequence $(\sum_{m=0}^{N} P^m f(0)(x_{n_k}))_{k=1}^{\infty}$ converges, there exists
for all \( j, k \geq k_0 \). It follows that
\[
|f(x_j) - f(x_k)| \leq 3\varepsilon
\]
for all \( j, k \geq k_0 \). Thus \((f(x_n))\) converges for every \( f \in \mathcal{H}(E) \). By (b) \((x_n)\) converges in \( E \), and \( A \) is relatively compact.

Remarks 1.2. (a) Sequences cannot be replaced by nets in condition (b) of Theorem 1.1. Let us see why. On the one hand the Banach space \( E = c_0(I) \) is weakly compactly generated for every index set \( I \) (see [13]), and hence every bounding subset of \( E \) is relatively compact. On the other hand, for each \( f \in \mathcal{H}(E) \) there is a countable set \( J_f \subseteq I \) such that \( f(x) = f(0) \) whenever \( \|x\|_{J_f} = 0 \) (see [10]). Now let \( \alpha \) be an uncountable ordinal, and let \( I = [0, \alpha] \) be the set of all ordinals that are smaller than or equal to \( \alpha \). Then the net of unit vectors \((e_\beta)_{\beta \in I} \in \mathcal{H}(E)\) converges to zero. Indeed for each \( f \in \mathcal{H}(E) \) there is \( \beta_f < \alpha \) such that \( J_f \subseteq [0, \beta_f] \). Hence \( \|e_\beta\|_{J_f} = 0 \), and therefore \( f(e_\beta) = f(0) \), for every \( \beta > \beta_f \).

(b) The technique used in proving the implication (b) \( \Rightarrow \) (a) in Theorem 1.1 also shows that every \( f \in \mathcal{H}(E) \) is weakly continuous on bounding sets. Indeed it suffices to recall that every continuous polynomial \( P : E \to \mathbb{C} \) is weakly continuous on bounding sets (see [7, theorem 3]), and then approximate \( f \) by its Taylor series.

2. Weak holomorphic convergence and limited sets

Now we deal with a question analogous to that in Section 1 for limited sets.

Proposition 2.1 [5]. Let \( (\varphi_n) \subseteq E' \). Then
\[
\sum_{n=1}^\infty (\varphi_n)^n \in \mathcal{H}(E) \text{ if and only if } \varphi_n \to 0 \text{ for } \sigma(E', E).
\]

Let \( \mathcal{D} = \{\sum_{n=1}^\infty (\varphi_n)^n : (\varphi_n) \subseteq E', \varphi_n \to 0 \text{ for } \sigma(E', E)\} \).

The next result is probably known, but since we did not find it in the literature, we include a proof for the convenience of the reader.

Proposition 2.2. A set \( A \subseteq E \) is limited if and only if every \( f \in \mathcal{D} \) is bounded on \( A \).

Proof. (\( \Rightarrow \)) Let \( f \in \mathcal{D} \), that is, \( f = \sum_{n=1}^\infty (\varphi_n)^n \), with \( (\varphi_n) \subseteq E', \varphi_n \to 0 \) for \( \sigma(E', E) \). If \( A \) is limited, then there exists \( N \in \mathbb{N} \) such that \( |\varphi_n(x)| \leq 1/2 \) for all \( x \in A \) and \( n \geq N \). Hence
\[
|f(x)| \leq \sum_{n=1}^{N-1} |\varphi_n(x)|^n + \sum_{n=N}^\infty 2^{-n}
\]
for every \( x \in A \). Since \( A \) is weakly bounded, it follows that \( f \) is bounded on \( A \).
If \( A \) is not limited, then we can find a \( \sigma(E',E) \)-null sequence \( (\varphi_n) \) in \( E' \) and a sequence \( (a_n) \) in \( A \) such that \(|\varphi_n(a_n)| > 2 \) for every \( n \in \mathbb{N} \). We can then inductively find a strictly increasing sequence \( (m_n) \subset \mathbb{N} \) such that

\[
|\varphi_n(a_n)|^{m_n} > n + 1 + \sum_{k<n} |\varphi_k(a_n)|^{m_k}
\]

for every \( n \in \mathbb{N} \). By Proposition 2.1 \( f = \sum_{n=1}^{\infty} (\varphi_n)^{m_n} \in \mathcal{H}(E) \) and \(|f(a_n)| > n \) for every \( n \in \mathbb{N} \). Thus \( f \in \mathcal{D} \) and \( f \) is unbounded on \( A \).

In view of Proposition 2.2, a straightforward adaptation of the proof of Theorem 1.1 yields the following theorem.

**Theorem 2.3.** For a Banach space \( E \) the following conditions are equivalent:

(a) every limited subset of \( E \) is relatively compact;

(b) a sequence \( (x_n) \) converges in \( E \) if and only if \( (f(x_n)) \) converges in \( C \) for every \( f \in \mathcal{D} \).

So far we have seen a certain analogy between bounding and limited sets. Our next proposition stresses that analogy. Here \( \tau_0 \) denotes the compact-open topology.

**Proposition 2.4.** A set \( A \subset E \) is bounding if and only if every null sequence in \( (\mathcal{H}(E), \tau_0) \) converges to zero uniformly on \( A \).

**Proof.** \((\Rightarrow)\) Since \( (f_n) \) is \( \tau_0 \)-bounded, it is uniformly bounded on \( 2A \) (see [16, example 12.C] or [17, proposition 2.5]). Since we may assume that \( A \) is balanced, an application of the Cauchy integral formula shows that for each \( \varepsilon > 0 \) there is \( N \in \mathbb{N} \) such that

\[
|f_n(x) - \sum_{m=0}^{N} P^m f_n(0)(x)| \leq \varepsilon
\]

for all \( x \in A \) and \( n \in \mathbb{N} \). Since \( f_n \to 0 \) in \( (\mathcal{H}(E), \tau_0) \), another application of the Cauchy integral formula shows that the sequence of polynomials \( \left( \sum_{m=0}^{N} P^m f_n(0) \right) \) converges to zero pointwise in \( E \), and thus converges to zero uniformly on \( A \), by [7, theorem 5]. Hence there is \( n_0 \in \mathbb{N} \) such that

\[
\left| \sum_{m=0}^{N} P^m f_n(0)(x) \right| \leq \varepsilon
\]

for all \( x \in A \) and \( n \geq n_0 \). It follows that \(|f_n(x)| \leq 2\varepsilon \) for all \( x \in A \) and \( n \geq n_0 \).

\((\Leftarrow)\) We first observe that the assumption implies that \( A \) is limited, and therefore bounded. And next we conclude that \( A \) is bounding by applying the assumption to the \( \tau_0 \)-null sequence \( (f - \sum_{m=0}^{N} P^m f(0))_{N=1}^{\infty} \) for each \( f \in \mathcal{H}(E) \).
References


