SPECTRAL MAPPING THEOREMS FOR HYPONORMAL OPERATORS

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Abstract

Let $T = H + iK$ be hyponormal and $\varphi$ be a strictly monotone increasing continuous function on $\sigma(H)$. We define $\tilde{\varphi}(T)$ by $\tilde{\varphi}(T) = \varphi(H) + iK$. In this paper, we show that if $z$ is an isolated eigenvalue of $\tilde{\varphi}(T)$, then the corresponding Riesz projection is self-adjoint. Also we introduce Xia spectrum and study the existence of an invariant subspace of an operator $\tilde{\varphi}(T)$.

1. Introduction

Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space $H$. An operator $T \in B(H)$ is said to be hyponormal if $T^*T \geq TT^*$.

For an operator $T$, we denote the positive part of $T$ by $|T|$ and decompose $T = H + iK$, with $H = \text{Re}T = \frac{1}{2}(T + T^*)$ and $K = \text{Im}T = \frac{i}{2}(T - T^*)$. Using this decomposition, several authors have given generalisations of hyponormal operators (see, e.g. [8; 9; 11; 13; 16]). Following Istrătescu [8; 9], in [3] we studied spectral properties of the operator $\tilde{\varphi}(T) = \varphi(H) + iK$ associated with a strictly monotone increasing continuous function $\varphi$ on the spectrum $\sigma(H)$. In the present paper, first we show that the Riesz projection for an isolated eigenvalue of $\tilde{\varphi}(T)$ is self-adjoint. In [1] we gave a condition for the existence of invariant subspaces in the case of $p$-hyponormal operations. Next we introduce Xia spectrum of $\tilde{\varphi}(T)$ and study the existence of invariant subspaces of an operator $\tilde{\varphi}(T)$. Also we study structural properties of the spectrum of $\tilde{\varphi}(T)$ for a pure hyponormal operator $T$.

Let $T \in B(H)$. We denote the spectrum, the point spectrum and the approximate point spectrum of $T$ by $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ respectively.

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A point $z \in \mathbb{C}$ is in the essential spectrum $\sigma_e(T)$ of $T$ if there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $x_n \rightarrow 0$ (weakly) and $(T - z)x_n \rightarrow 0$. It is well known that

$$\sigma(T) = \sigma_e(T) \cup \sigma_p(T) \cup \{z : z \in \sigma_p(T^*)\}, \quad (1)$$

(see [7]). Throughout this paper, for a strictly monotone increasing continuous function $\varphi(\cdot)$ on $\mathbb{R}$ we define $\tilde{\varphi}(\cdot)$ on $\mathbb{C}$ by $\tilde{\varphi}(a + ib) = \varphi(a) + ib$ for $a, b \in \mathbb{R}$.

For a pair of operators $(T_1, T_2)$, $(z_1, z_2) \in \mathbb{C}^2$ is in the joint approximate point spectrum $\sigma_a(T_1, T_2)$ if there exists a sequence $\{x_n\}$ of unit vectors such that

$$\| (T_j - z_j)x_n \| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \quad (j = 1, 2).$$

We denote the (Taylor) joint spectrum of a commuting pair of operators $(T_1, T_2)$ by $\sigma(T_1, T_2)$ (see [14]).

2. Theorems

In this paper we sometimes use the following results: let $T = H + iK$ be hyponormal and $\varphi(\cdot)$ be a strictly monotone increasing continuous function on $\sigma(H)$.

(1) If $\tilde{\varphi}(T) - (a + ib)x_n \rightarrow 0$, then there exists $a_0$ such that $\varphi(a_0) = a$, $(H - a_0)x_n \rightarrow 0$, $(K - b)x_n \rightarrow 0$ and $\tilde{\varphi}(T) - (a + ib))^nx_n \rightarrow 0$ [3, lemma 2].

(2) $\sigma(\tilde{\varphi}(T)) = \tilde{\varphi}(\sigma(T))$ and $\sigma_a(\tilde{\varphi}(T)) = \tilde{\varphi}(\sigma_a(T))$ [3, theorem 1].

First we show the following:

**Theorem 1.** Let $T = H + iK$ be hyponormal and $\varphi(\cdot)$ be a strictly monotone increasing continuous function on $\sigma(H)$. We define $\tilde{\varphi}(\cdot)$ by $\tilde{\varphi}(a + ib) = \varphi(a) + ib$ and $\tilde{\varphi}(T) = \varphi(H) + iK$. Let $z$ be an isolated eigenvalue of $\tilde{\varphi}(T)$ and $r > 0$ be $\lambda : |\lambda - z| \leq r \cap \sigma(\tilde{\varphi}(T)) = \{z\}$. Put $E = \frac{1}{2\pi i} \int_{|\lambda - z| = r} (\lambda - \tilde{\varphi}(T))^{-1}d\lambda$. Then the following statements hold:

1. $E(\mathcal{H}) = \ker(\tilde{\varphi}(T) - z) = \ker((\tilde{\varphi}(T) - z)^*)$.
2. $E$ is self-adjoint.

**Proof.** (1) We show $\ker(\tilde{\varphi}(T) - z) = \ker((\tilde{\varphi}(T) - z)^*)$. Let $\tilde{\varphi}(z_0) = z$. Then $z_0$ is an isolated eigenvalue of $T$ and by [3, corollary 4] we have $\ker(\tilde{\varphi}(T) - z) = \ker(T - z_0)$, hence $\ker(T - z_0)$ reduces for $T$ and $\ker(\tilde{\varphi}(T) - z)$ reduces for $\tilde{\varphi}(T)$. Therefore, we have the decomposition

$$\tilde{\varphi}(T) = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \quad \text{on} \quad \ker(\tilde{\varphi}(T) - z) \oplus \ker(\tilde{\varphi}(T) - z)^{\perp}. $$

We may consider that this decomposition is on $\ker(T - z_0) \oplus \ker(T - z_0)^{\perp}$. Let $N = \ker(T - z_0)$. Since $T_{\perp V} = H_{\perp V} + iK_{\perp V}$ (hyponormal operator) and $T_{\perp V} = H_{\perp V} + iK_{\perp V}$ (hyponormal operator) are Cartesian decompositions, hence we have

$$\tilde{\varphi}(T_{\perp V}) = \varphi(H_{\perp V}) = iK_{\perp V} \quad \text{and} \quad \tilde{\varphi}(T_{\perp V}) = \varphi(H_{\perp V}) + iK_{\perp V}.$$

Also we have $\tilde{\varphi}(T_{\perp V}) - z = A$. If $0 \notin \sigma(A)$, then $z \in \sigma(\tilde{\varphi}(T_{\perp V}))$, and hence $z$ is an isolated point of $\sigma(\tilde{\varphi}(T_{\perp V}))$. Therefore $z$ is an eigenvalue of $\tilde{\varphi}(T_{\perp V})$. This is a contradiction. Since $A$ is invertible, we have $\ker(\tilde{\varphi}(T) - z) = \ker((\tilde{\varphi}(T) - z)^*)$. Hence we have $E(\mathcal{H}) = \ker(\tilde{\varphi}(T) - z) = \ker((\tilde{\varphi}(T) - z)^*)$. 


(2) Since \(\{\lambda : |\lambda - z| \leq r\} \cap \sigma(A) = \emptyset\), it holds that \(\frac{1}{2\pi i} \int_{|\lambda - z| = r} (\lambda - z - A)^{-1} d\lambda = 0\). Hence we have

\[
E = \frac{1}{2\pi i} \int_{|\lambda - z| = r} (\lambda - \bar{\varphi}(T))^{-1} d\lambda = \frac{1}{2\pi i} \int_{|\lambda - z| = r} \begin{pmatrix}
(\lambda - z)^{-1} & 0 \\
0 & (\lambda - z - A)^{-1}
\end{pmatrix} d\lambda
\]

\[
= \begin{pmatrix}
\frac{1}{2\pi i} \int_{|\lambda - z| = r} (\lambda - z)^{-1} d\lambda & 0 \\
0 & \frac{1}{2\pi i} \int_{|\lambda - z| = r} (\lambda - z - A)^{-1} d\lambda
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Therefore, \(E\) is self-adjoint. So the proof is complete.

In [2] we introduced Xia spectrum for a \(p\)-hyponormal operator. For the existence of invariant subspaces of \(\bar{\varphi}(T)\), we introduce Xia spectrum of \(\bar{\varphi}(T)\). Let \(T = H + iK\) be hyponormal. Define

\[
\mathcal{S}^+(K) = \lim_{t \to \infty} e^{itH}Ke^{-itH} \quad \text{and} \quad \mathcal{S}^-(K) = \lim_{t \to \infty} e^{-itH}Ke^{itH}.
\]

If \(T = H + iK\) is hyponormal, then there exist operators \(\mathcal{S}^\pm(K)\) (see [15]). Define

\[
K_k = k\mathcal{S}^+(K) + (1-k)\mathcal{S}^-(K),
\]

for \(k \in [0, 1]\). Then we have the following:

**Theorem A.** [4, lemma 2.1]. Under the above notations, if \(T = H + iK\) is hyponormal, then \(HK_k = K_kH\) and \(\phi(H)K_k = K_k\phi(H)\) for every \(k \in [0, 1]\).

Therefore, it is clear that \(HK_k = K_kH\) and \(\phi(H)K_k = K_k\phi(H)\) for every \(k \in [0, 1]\). We define

\[
T_k = H + iK_k \quad \text{and} \quad \bar{\varphi}(T_k) = \varphi(H) + iK_k \quad \text{for} \; k \in [0, 1].
\]

Then, for \(k \in [0, 1]\), \(T_k\) and \(\bar{\varphi}(T_k)\) are normal and the following theorem holds.

**Theorem B.** [16, theorem IV.3.1]. Let \(T = H + iK\) be hyponormal. Then

\[
\sigma(T) = \bigcup_{0 \leq k \leq 1} \sigma(T_k).
\]

**Definition 1.** We define Xia spectrum \(\sigma_X(\bar{\varphi}(T))\) of \(\bar{\varphi}(T)\) by

\[
\sigma_X(\bar{\varphi}(T)) = \bigcup_{0 \leq k \leq 1} \sigma_d(\varphi(H), K_k).
\]

**Remark.** We cannot define Xia spectrum for an arbitrary operator. We prepare the following results.

**Theorem C.** [6]. Let \((T_1, T_2)\) be a commuting pair of normal operators. Then

\[
\sigma(T_1, T_2) = \sigma(a(T_1, T_2)).
\]
Theorem D. [5; 12; 14]. Let \((T_1, T_2)\) be a commuting pair and \(f(z_1, z_2)\) be a polynomial of two variables. Then
\[
f(\sigma(T_1, T_2)) = \sigma(f(T_1, T_2)) \quad \text{and} \quad f(\sigma_a(T_1, T_2)) = \sigma_a(f(T_1, T_2)).
\]

Theorem 2. Let \(T = H + iK\) be hyponormal and \(\varphi(\cdot)\) be a strictly monotone increasing continuous function on \(\sigma(H)\). Then
\[
a + ib \in \sigma(\varphi(T)) \quad \text{if and only if} \quad (a, b) \in \sigma_X(\varphi(T)).
\]

Proof. Let \(a + ib \in \sigma(\varphi(T))\). Then there exists \(a_0\) such that \(a_0 + ib \in \sigma(T)\) and \(\varphi(a_0) = a\). Since \(T = H + iK\) is hyponormal, there exists \(k \leq 1\) such that \((a_0, b) \in \sigma_a(H, K_k)\). Hence we have \((a, b) \in \sigma_a(\varphi(H), K_k)\) and \((a, b) \in \sigma_X(\varphi(T))\). Next assume that \((a, b) \in \sigma_X(\varphi(T))\). Then there exists \(k \leq 1\) such that \((a, b) \in \sigma_a(\varphi(H), K_k)\). Since \(T = H + iK\) is a normal operator, \(\varphi(T_k) = \varphi(H) + iK_k\) and \(a + ib \in \sigma_a(\varphi(T))\), and by lemma 2 of [3] there exists \((a_0, b) \in \sigma_a(H, K_k)\) such that \(\varphi(a_0) = a\). Since \(a_0 + ib \in \sigma(T)\), by theorem 1 of [3] we have \(a + ib \in \sigma(\varphi(T))\). So the proof is complete. □

Corollary 3. Let \(T = H + iK\) be hyponormal and \(\varphi(\cdot)\) be a strictly monotone increasing continuous function on \(\sigma(H)\). Then
\[
\sigma(\varphi(T)) = \bigcup_{0 \leq k \leq 1} \sigma(\varphi(T_k)).
\]

Proof. Let \(a + ib \in \sigma(\varphi(T))\). Then by Theorem 2 there exists \(k \leq 1\) such that \((a, b) \in \sigma_a(\varphi(H), K_k)\). Hence by Theorem D it is clear that \(a + ib \in \sigma(\varphi(T_k))\). Hence \(a + ib \in \bigcup_{0 \leq k \leq 1} \sigma(\varphi(T_k))\). Next let \(a + ib \in \sigma(\varphi(T_k))\) for some \(0 \leq k \leq 1\). Since \(\varphi(T_k)\) is a normal operator, we have \(a + ib \in \sigma_a(\varphi(T_k))\). Since \(\varphi(H), K_k\) is a commuting pair of hermitian operators, by Theorem C we have \((a, b) \in \sigma_a(\varphi(H), K_k)\). Hence by Theorem 2 we have \(a + ib \in \sigma(\varphi(T))\). □

We introduce the generalised spectrum \(G(|\varphi(T)|)\) of \(|\varphi(T)|\).

Definition 2. Let \(T = H + iK\) be hyponormal and \(\varphi(\cdot)\) be a strictly monotone increasing continuous function on \(\sigma(H)\). For \(k \in [0, 1]\), let \(T_k\) be the operator defined following Theorem A. For an operator \(\varphi(T)\), the generalised spectrum \(G(|\varphi(T)|)\) of \(|\varphi(T)|\) is defined by
\[
G(|\varphi(T)|) = \bigcup_{0 \leq k \leq 1} \sigma(\varphi(T_k)).
\]

Theorem 4. Let \(T = H + iK\) be hyponormal and \(\varphi(\cdot)\) be a strictly monotone increasing continuous function on \(\sigma(H)\).
\begin{enumerate}
\item If \(r \in G(|\varphi(T)|)\), then there exists \(a + ib \in \sigma(\varphi(T))\) such that \(|a + ib| = r\).
\item If \(z = a + ib \in \sigma(\varphi(T))\), then \(|z| \in G(|\varphi(T)|)\).
\end{enumerate}
Moreover

PROOF. By the assumption, there exist non-negative numbers in an invariant subspace

continuous function on

continuous function on

PROOF. If \( T = H + iK \) be hyponormal and \( \varphi(\cdot) \) be a strictly monotone increasing continuous function on \( \sigma(H) \). If \( G(\varphi(T)) \) is not interval, then \( \tilde{\varphi}(T) \) has a non-trivial invariant subspace.

PROOF. By the assumption, there exist non-negative numbers \( r_1 \) and \( r_2 \) such that \( r_1 < r_2 \) and \( (r_1, r_2) \cap G(\varphi(T)) = \emptyset \). By theorem 4(1) there exist \( z_1 \) and \( z_2 \) in \( \sigma(\varphi(T)) \) such that

Thus \( \sigma(\tilde{\varphi}(T)) \) is not connected and \( \tilde{\varphi}(T) \) has a non-trivial invariant subspace.

The boundary of a set \( E \subset \mathbb{C} \) is denoted by \( \partial E \). An operator \( T \) is called pure hyponormal if it has no non-trivial reducing subspace on which it is normal.

Theorem 5. Let \( T = H + iK \) be hyponormal and \( \varphi(\cdot) \) be a strictly monotone increasing continuous function on \( \sigma(H) \). If \( G(\varphi(T)) \) is not interval, then \( \tilde{\varphi}(T) \) has a non-trivial invariant subspace.

PROOF. By the assumption, there exist non-negative numbers \( r_1 \) and \( r_2 \) such that \( r_1 < r_2 \) and \( (r_1, r_2) \cap G(\varphi(T)) = \emptyset \). By theorem 4(1) there exist \( z_1 \) and \( z_2 \) in \( \sigma(\varphi(T)) \) such that

Thus \( \sigma(\tilde{\varphi}(T)) \) is not connected and \( \tilde{\varphi}(T) \) has a non-trivial invariant subspace.

The boundary of a set \( E \subset \mathbb{C} \) is denoted by \( \partial E \). An operator \( T \) is called pure hyponormal if it has no non-trivial reducing subspace on which it is normal.

Theorem 6. Let \( T = H + iK \) be hyponormal and \( \varphi(\cdot) \) be a strictly monotone increasing continuous function on \( \sigma(H) \). If \( z \in \partial \sigma(\tilde{\varphi}(T)) \), then

Moreover, if \( T \) is pure, then

PROOF. Since \( z \in \partial \sigma(\tilde{\varphi}(T)) \subset \sigma_d(\tilde{\varphi}(T)) \), we have \( z \in \sigma_d(\tilde{\varphi}(T)) \) by lemma 2 of [3]. Hence we have \( |z| \in \sigma(\tilde{\varphi}(T)) \cap \sigma(\tilde{\varphi}(T)^*) \).

Next we assume that \( T \) is pure. By theorem 1 of [3], let \( z_0 \) be \( \tilde{\varphi}(z_0) = z \). Since \( \varphi \) is strictly monotone increasing, we have \( z_0 \in \partial \sigma(T) \). Hence by theorem 1 of [11] there exists a sequence \( \{x_n\} \) of unit vectors such that \( x_n \rightarrow 0 \) (weakly) and \( (T - z)x_n \rightarrow 0 \). Since \( T \) is hyponormal, it is easy to see that \( (\tilde{\varphi}(T) - z)x_n \rightarrow 0 \) and \( (\tilde{\varphi}(T) - z)^*x_n \rightarrow 0 \). Therefore, we have

So the proof is complete.

Theorem 7. Let \( T = H + iK \) be hyponormal and \( \varphi(\cdot) \) be a strictly monotone increasing continuous function on \( \sigma(H) \). If \( z \in \sigma(\tilde{\varphi}(T)) \) and \( \tilde{z} \notin \sigma_p(\tilde{\varphi}(T)^*) \), then \( |z| \in \sigma_c(\tilde{\varphi}(T)) \cap \sigma_c(\tilde{\varphi}(T)^*) \).

PROOF. If \( z \in \sigma_p(\tilde{\varphi}(T)) \), then by lemma 2 of [3] we have \( z \in \sigma_p(\tilde{\varphi}(T)^*) \). Hence we have \( z \notin \sigma_p(\tilde{\varphi}(T)) \). By Equation (1) we have \( z \in \sigma_c(\tilde{\varphi}(T)) \). Hence there exists a sequence \( \{x_n\} \) of
unit vectors such that
\[ x_n \rightarrow 0 \text{ (weakly)} \quad \text{and} \quad (\tilde{\varphi}(T) - z)x_n \rightarrow 0. \]
By lemma 2 of [3] it holds that 
\[ (|\tilde{\varphi}(T)| - |z|)x_n \rightarrow 0 \quad \text{and} \quad (|\tilde{\varphi}(T)|^* - |z|)x_n \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]
Since \( x_n \rightarrow 0 \) (weakly), we have \( |z| \in \sigma_e(|\tilde{\varphi}(T)|) \cap \sigma_e(|\tilde{\varphi}(T)|^*) \). So the proof is complete. \( \blacksquare \)

In order to give a proof of Theorem 8, we need the following simple lemma. We contain its proof for the sake of completeness. Let \( m_1 \) and \( m_2 \) be the Lebesgue measure on \( \mathbb{R} \) and the planar Lebesgue measure, respectively.

**Lemma 1.** Let \( \varphi(\cdot) \) be a strictly monotone increasing continuous function on \( \mathbb{R} \) such that \( \varphi^{-1} \) is absolutely continuous. Let \( A \) be a Lebesgue measurable set in \( \mathbb{R} \). If \( m_2(\varphi^{-1}(A)) = 0 \), then \( m_2(A) = 0 \).

**Proof.** Recall \( \tilde{\varphi}(x, y) = (\varphi(x), y) \). For \( E \subset \mathbb{R}^2 \) and \( y \in \mathbb{R} \), we define \( E_y = \{x : (x, y) \in E\} \). We have \( \varphi^{-1}(A)_y = \tilde{\varphi}^{-1}(A_y) \), so that
\[
\begin{align*}
m_2(\varphi^{-1}(A)) = & \int_{\mathbb{R}} m_1(\tilde{\varphi}^{-1}(A)_y) \, dm_1(y) = \int_{\mathbb{R}} m_1(\tilde{\varphi}^{-1}(A_y)) \, dm_1(y) = 0.
\end{align*}
\]
Then
\[
m_1(\varphi^{-1}(A)_y) = 0 \quad \text{for a.e.} \quad y.
\]
Since \( \varphi^{-1} \) is absolutely continuous,
\[
m_1(A_y) = 0 \quad \text{for a.e.} \quad y.
\]
Therefore,
\[
m_2(A) = \int_{\mathbb{R}} m_1(A_y) \, dm_1(y) = 0.
\]
So the proof is complete. \( \blacksquare \)

**Theorem 8.** Let \( T = H + iK \) be pure hyponormal and \( \varphi(\cdot) \) be a strictly monotone increasing continuous function on \( \mathbb{R} \) such that \( \varphi^{-1} \) is absolutely continuous. If \( G \) is any open disk of the plane, then \( m_2(\sigma(\tilde{\varphi}(T)) \cap G) > 0 \) whenever \( \sigma(\tilde{\varphi}(T)) \cap G \neq \emptyset \).

**Proof.** Assume that, for some open disk \( G \), \( \sigma(\tilde{\varphi}(T)) \cap G \neq \emptyset \) and \( m_2(\sigma(\tilde{\varphi}(T)) \cap G) = 0 \). Let \( z \in \sigma(\tilde{\varphi}(T)) \cap G \). Then there exists \( z_0 \in \sigma(T) \) such that \( \tilde{\varphi}(z_0) = z \). Since \( \tilde{\varphi} \) is continuous, there exists an open disk \( G_0 \) such that
\[
\begin{align*}
z_0 \in G_0 \quad \text{and} \quad \tilde{\varphi}(G_0) \subset G.
\end{align*}
\]
By Lemma 1 it holds that \( m_2(\sigma(T) \cap \tilde{\varphi}^{-1}(G)) = 0 \). Since \( \sigma(T) \cap G_0 \subset \sigma(T) \cap \tilde{\varphi}^{-1}(G) \), so that \( m_2(\sigma(T) \cap G_0) = 0 \). But \( z_0 \in \sigma(T) \cap G_0 \). Since \( T \) is a pure hyponormal operator, this is a contradiction to Putnam’s result [10, theorem 4]. Hence the proof is complete. \( \blacksquare \)
Finally we show the following:

**Theorem 9.** Let $T=H+iK$ be pure hyponormal and $\varphi(\cdot)$ be a strictly monotone increasing continuous function on $\sigma(H)$. If $m_1(\sigma(\varphi^{-1}(T)))=0$, then there exists a finite or countably infinite number of pairwise disjoint open annuli $A_n=\{z:a_n<|z|<b_n\}$ ($n=1,2,\ldots$) such that $\sigma(\varphi(T))$ is the closure of the set $\bigcup A_n$ and $\bigcup A_n\subset\sigma_p(\varphi(T)^*)$.

**Proof.** Let $z_1\in\sigma(\varphi(T))$. Consider any open disk $G$ containing $z_1$. Then $G$ contains a closed disk $F$ satisfying

$$F=\{z:|z-z_1|\leq\delta, \delta>0\}\subset\sigma(\varphi(T)).$$

Otherwise $\sigma(\varphi(T))\cap G\subset\partial\sigma(\varphi(T))$. Hence if the half-line $\ell:\theta=constant$ intersects $G$, then by Theorem 6 each $r$ satisfying $re^{i\theta}\in\ell\cap(\sigma(\varphi(T))\cap G)$ belongs to $\sigma(\varphi(T))$. Hence, by this assumption the set of such numbers $r$ has linear measure 0. It follows from Fubini’s theorem that $m_2(\sigma(\varphi(T))\cap G)=0$. By Theorem 8, this is a contradiction. This proves (2).

Let $a=\inf\{|z|:z\in F\}$ and $b=\sup\{|z|:z\in F\}$. Then $a<b$ and

$$\{z:a\leq|z|\leq b\}\subset\sigma(\varphi(T)).$$

In fact, assume that there exists $z_2\notin\sigma(\varphi(T))$ such that $a<|z_2|<b$. Since $z_2\in\sigma(\varphi(T))^C$, there exists $\varepsilon>0$ such that

$$\{z:|z-z_2|<\varepsilon\}\subset\sigma(\varphi(T))^C.$$ 

We may assume that $\varepsilon<\min\{|z_2|-a, b-|z_2|\}$. Hence by Theorem 6 we have $[|z_2|-\varepsilon, |z_2|]\subset\sigma(\varphi(T))$ or $[|z_2|, |z_2|+\varepsilon]\subset\sigma(\varphi(T))$. This is a contradiction to $m_1(\sigma(\varphi(T)))=0$. Hence it follows (3). Therefore, we can see that $\sigma(\varphi(T))$ can be taken as the closure of countably union of disjoint closed annuli, each of the form $\{z:c\leq|z|\leq d\}$ with $c<d$.

For a fixed annulus let $\cup(c_n, d_n)$ denote the canonical decomposition of the linear open set $[c, d]-\{[c, d]\cap\sigma(\varphi(T)^*)\}$. In fact for $z$ in this open set, if $|z|=d$, then $z\in\partial\sigma(\varphi(T))$. Hence $z\in\sigma(\varphi(T)^*)$. Also, if $|z|=c$, then we have $|z|\in\sigma(\varphi(T)^*)$. It follows that $\{z:c\leq|z|\leq d\}$ is the closure of $\cup B_n$, where $B_n=\{z:c_n<|z|<d_n\}$. Also by Theorem 7 we have each $B_n\subset\sigma_p(\varphi(T)^*)$. This completes the proof.

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**References**


