SILVA DISTRIBUTIONS FOR CERTAIN LOCALLY COMPACT GROUPS

SUZANA METELLO DE NÁPOLES

We define axiomatically a space of distributions or generalized functions for a class of locally compact groups, the parametrizable groups (cf. Th. 1), using a method inspired in the papers by J.S. Silva concerning distributions on $\mathbb{R}^n$ (cf. [6]).

We establish an isomorphism between the space we define and the space of distributions by Bruhat (cf. [1]).

1 – Notations and preliminary results

Let $G$ be a locally compact abelian and connected group. There exists an indexed family $F = \{K_\alpha : \alpha \in A\}$ of compact subgroups of $G$, such that $\bigcap_{\alpha \in A} K_\alpha = \{0\}$, $K_\alpha \cap K_\beta \in F$, $\forall \alpha, \beta \in A$ and each $G/K_\alpha$ is a Lie group $L_\alpha$. We set $\alpha < \beta \Leftrightarrow K_\alpha \supset K_\beta$.

Let $\phi_\alpha$ and $\phi_{\alpha \beta}$ be the canonical homomorphisms, $\phi_\alpha : G \to G/K_\alpha$, $\phi_{\alpha \beta} : G/K_\beta \to G/K_\alpha$. The projective limit of the projective system $(L_\alpha, \phi_{\alpha \beta})_{\alpha, \beta \in A}$ is identical with $G$ and the canonical map from this limit onto each $L_\alpha$ is identical with $\phi_\alpha$.

Note that if we take $\alpha_0 \in A$ and consider the subset $A_0 = \{\alpha \in A : \alpha > \alpha_0\}$ we have $G = \lim_{\alpha \in A} L_\alpha = \lim_{\alpha \in A_0} L_\alpha$. So, without less generality, we can suppose that the set $A$ has a first element $\alpha_0$.

The continuous homomorphisms $\phi_\alpha$ and $\phi_{\alpha \beta}$ are open and proper.

If we suppose that $G$ is metrizable, we have $G = \lim_{i \in \mathbb{N}_0} L_i$.

As $G$ is locally compact and connected, $G = \bigcup_{j \in \mathbb{N}} W_j$, where each $W_j$ is an open subset of $G$ which is relatively compact and such that $\overline{W}_j \subset W_{j+1}$ ($j \in \mathbb{N}$).

Received: May 16, 1992.
If $B$ is any compact subset of $G$, there exists $j \in \mathbb{N}$ such that $B \subset W_j$.

For each $j \in \mathbb{N}$ we set $\Omega_j = K_{\alpha} + W_j$ and $\Delta_j = K_{\alpha} + \overline{W_j}$. We obtain a sequence of compact subsets of $G$ such that, if $B$ is any compact subset of $G$, $B \subset \Delta_j$ for a certain $j \in \mathbb{N}$. We also have $\phi_{\alpha}^{-1}(\phi_{\alpha}(\Delta_j)) = \Delta_j$ for each $\alpha \in A$ and each $j \in \mathbb{N}$.

We use the symbol $\mathcal{G}$ to denote the Lie algebra of $G$ (cf. [2]) that is $\mathcal{G} = \varprojlim \mathcal{L}_\alpha$, each $\mathcal{L}_\alpha$ denoting the Lie algebra of $L_\alpha$. The canonical homomorphisms $d\phi_\alpha : \mathcal{G} \to \mathcal{L}_\alpha$ and $d\phi_{\alpha,\beta} : \mathcal{L}_\beta \to \mathcal{L}_\alpha$ are onto and induced by $\phi_\alpha$ and $\phi_{\alpha,\beta}$. The exponential map $\exp : \mathcal{G} \to G$ is defined by $\exp(X_\alpha)_{\alpha \in A} = (\exp X_\alpha)_{\alpha \in A}$.

We denote by $n_\alpha$ the dimensions of each $L_\alpha$. If $\alpha < \beta$ we have $n_\alpha \leq n_\beta$.

The exponential exp is an analytic diffeomorphic map from an open neighborhood $B_\alpha$ of zero in $\mathcal{L}_\alpha$ onto an open neighborhood $V_\alpha$ of zero in $L_\alpha$. Let $U_\alpha \subset V_\alpha$ be such that $U_\alpha + U_\alpha \subset V_\alpha$; we consider the canonical identification of $\mathcal{L}_\alpha$ with $\mathbb{R}^{n_\alpha}$ and we call the couple $(\xi_\alpha, U_\alpha)$ with $\xi_\alpha = (\exp |V_\alpha|^{-1})_{U_\alpha}$, the canonical chart of $L_\alpha$.

For each $a_\alpha \in L_\alpha$ we set $U_{a_\alpha} = a_\alpha + U_\alpha$ and $\xi_{a_\alpha}(u_\alpha) = \xi_\alpha(u_\alpha - a_\alpha)$. We call canonical atlas of $L_\alpha$ to $\{(\xi_{a_\alpha}, U_{a_\alpha}) : a_\alpha \in L_\alpha\}$.

Let $\mathcal{Q}_\alpha = \{(t_i) ; t_i \in \mathbb{R}^{n_\alpha} : |t_i| < \delta_i\}$, we call $V_{a_\alpha} = a_\alpha + \xi_{a_\alpha}^{-1}(\mathcal{Q}_\alpha)$ an elementary neighborhood of $a_\alpha \in L_\alpha$.

For each $p = (p_i) \in \mathbb{N}_0^{n_\alpha}$, let $N^p(\mathcal{Q}_\alpha)$ be the subspace of the space $C(\mathcal{Q}_\alpha)$ of continuous complex-valued functions defined in $\mathcal{Q}_\alpha$ of the type

$$\theta_\alpha(t_1, \ldots, t_{n_\alpha}) = \sum_{i=1}^{n_\alpha} \sum_{k=0}^{p_i-1} \sum_{h=0} \ h_{ik}(t_1, \ldots, t_{n_\alpha})$$

with $h_{ik} \in C(\mathcal{Q}_\alpha)$ and independent of $t_i$. The elements of $N^p(\mathcal{Q}_\alpha)$ are called pseudo-polynomials in $\mathcal{Q}_\alpha$ with degree less then $p$.

A function $f_\alpha \in C(V_{a_\alpha})$ such that $f \circ \exp \in N^p(\xi_\alpha(V_{a_\alpha} - a_\alpha))$ is called a pseudo-polynomial on $V_{a_\alpha}$ with degree less then $p$.

We call fundamental family of one parameter subgroups of $L_\alpha$, the family of parameter subgroups determined by the elements of one basis $B_\alpha$ of $\mathcal{L}_\alpha$, that is, $x_\alpha(t) = \exp tX_\alpha, t \in \mathbb{R}$, $X_\alpha \in B_\alpha$. Each basis $B_\alpha$ of $\mathcal{L}_\alpha$ determines a fundamental family $S_\alpha$ of one parameter subgroups of $L_\alpha$.

Note that we can write $V_{a_\alpha} = \{a_\alpha + \sum_{i=1}^{n_\alpha} x_{i,a}(t_i) \cup x_{i,a} \in S_\alpha, (t_i) \in \mathcal{Q}_\alpha\}$.

**Theorem 1.** Let $G$ be an abelian locally compact group, connected and metrizable, $G = \varprojlim L_i$. There exists a family $S$ of one parameter subgroups of
\( G \) and fundamental families \( S_i \) of one parameter subgroups of \( L_i \) \((i \in \mathbb{N}_0)\) such that, \( \forall x \in S, \phi_i \circ x = x_i \in S_i \) or \( \phi_i \circ x = 0 \), each \( x_i \in S_i \) being obtained as the image of just one element \( x \in S \).

**Proof:** As \( \ker d\phi_i \supset \ker d\phi_{i+1} \) \((i \in \mathbb{N}_0)\), we consider \( H_i \) such that \( H_i \oplus \ker d\phi_i = \ker d\phi_0 \). We have \( H_i \subset H_{i+1} \) \((i \in \mathbb{N}_0)\).

We choose basis \( B_i^* \) of each \( H_i \) such that \( B_i^* \subset B_{i+1}^* \): we have \( d\phi_i(\ker d\phi_0) = \ker d\phi_{0i} \), so \( B_{0i} = d\phi_i(B_i^*) \) is a basis for \( \ker d\phi_0 \) \((i \in \mathbb{N})\).

If we take \( H_0 \) such that \( H_0 \oplus \ker d\phi_0 = \mathcal{G} \) and a basis \( B_0^* \) of \( H_0 \) then \( B_0 = d\phi_0(B_0^*) \) is a basis of \( L_0 \). We obtain a sequence of basis \((B_i)\) of \((L_i)\) setting \( B_0 = d\phi_0(B_0^*) \) and \( B_i = d\phi_i(B_0^*) \cup B_{0i}, \forall i \in \mathbb{N} \): we have \( d\phi_{ij}(B_j) = B_i \cup \{0\} \) \((i, j \in \mathbb{N}_0, i < j)\).

Let’s take \( B_i = B_i \cup \{0\} \) \((i \in \mathbb{N}_0)\) and \( B = \bigcup_i B_i^* \). We have that \( d\phi_i(B) = B_i \) \((i \in \mathbb{N}_0)\), each \( X_i \in B_i \) is obtained as the image of just one element \( X \in B \) and \( B = \lim_{i \in \mathbb{N}_0} B_i \).

We complete our proof by taking \( S = \{x(t) = \exp tx : X \in B\} \) and \( S_i = \{x_i(t) = \exp tx_i : X_i \in B_i\} \).

**Note:** The construction of the family \( S \), related with well chosen families \( S_i \) in the way expressed in the precedent theorem, can be done under more general hypothesis. In fact let \( G = T^J \times \mathbb{R}^p \) with \( p \in \mathbb{N} \) and \( J \) any set of indexes: if we take the family \((N_\alpha)\) of the finite parts of \( J \) and if we set \( \alpha < \beta \Leftrightarrow N_\alpha \subset N_\beta \), we have \( G = \lim \langle T^{N_\alpha} \times \mathbb{R}^p \rangle \). In \( \mathbb{R}^p \) we take, as usually, \( p \) one parameter subgroups

\[
y_1(t) = (t, 0, \ldots, 0), \ldots, y_p(t) = (0, \ldots, 0, t) \text{ and for each } j \in J \text{ and each } N_\alpha \text{ we set } z_{j,\alpha}(t) = (e^{ik_j t})_{k \in N_\alpha} \text{ and } z_j(t) = (e^{ik_j t})_{k \in J}. \]

The families

\[
S_\alpha = \{(z_{j,\alpha}, 0)\}_{j \in N_\alpha} \cup \{(0, y_m)\}_{1 \leq m \leq p}
\]

and

\[
S = \{(z_j, 0)\}_{j \in J} \cup \{(0, y_m)\}_{1 \leq m \leq p},
\]

with \((z_{j,\alpha}, 0)(t) = (z_{j,\alpha}(t), 0), (z_j, 0)(t) = (z_j(t), 0), \) and \((0, y_m)(t) = (1, y_m(t))\) verify the statement of our theorem.

We call parametrizable groups the abelian locally compact and connected groups \( G = \lim_{\alpha \in A} L_\alpha \) for which we can find families \( S \) and \( S_\alpha \) satisfying the conditions expressed on Theorem 1. We call the family \( S \) a fundamental family of one parameter subgroups of \( G \).

Now on we are only going to deal with parametrizable groups (in particular with metrizable groups).
Let $G = \lim_{\alpha \in A} L_\alpha$ be a parametrizable group, $S = \{x_i : i \in I\}$ a fundamental family of one parameter subgroups of $G$, \( \Omega \) an open set in $G$ and $f \in C(\Omega)$: $f$ is partially differentiable with respect to $x_i$ in $\Omega$ if for every $a \in \Omega$, the real-valued function $t \to f(a + x_i(t))$ is differentiable in a neighborhood of $0 \in \mathbb{R}$.

We set

$$D_{x_i} f(a) = \frac{d}{dt} f(a + x_i(t)) \bigg|_{t=0}$$

an more generally

$$D_{x_i} f(a + x_i(t)) = \frac{d}{dt} f(a + x_i(t)).$$

If $D_{x_i} D_{x_j} f$ and $D_{x_j} D_{x_i} f$ exists in $C(\Omega)$ we have $D_{x_i} D_{x_j} f = D_{x_j} D_{x_i} f$.

If $p = (p_i) \in \mathbb{N}_0^I$, we set $D^p = \prod_{i \in I} D_{x_i}^{p_i}$.

If $G$ is a lie group $L$, the partial-differentiation operators $D_{x_i}$ coincides with the usual operators respecting to a fundamental family of one parameter subgroups of $L$.

Let $\Delta$ be the closure of an open relatively compact subset $\Omega$ of $G$, $f \in C(\Delta)$ such that $D_{x_i}(f|_\Omega)$ can be continuously extended to $\Delta$. We also use the symbol $D_{x_i} f$ to denote the continuous function in $\Delta$ whose restriction to $\Omega$ coincides with $D_{x_i}(f|_\Omega)$.

We denote by $E(G)$ the subspace of $C(G)$ formed by the functions having the following property: for each $a \in G$ there exists $\alpha \in A$ and a function $f_\alpha$ continuous on a neighborhood $V_\alpha$ of $a_\alpha = \phi_\alpha(a)$ such that $f = f_\alpha \circ \phi_\alpha$ in $V_\alpha = \phi_\alpha^{-1}(V_{\alpha')}$.

If $\Delta$ is the closure of an open relatively compact subset of $G$, we denote by $E(\Delta)$ the subspace of $C(\Delta)$ formed by the functions that admits in $\Delta$, for a certain $\alpha \in A$, a decomposition of the type $f = f_\alpha \circ \phi_\alpha$ with $f_\alpha \in C(\phi_\alpha(\Delta))$.

We take on $E(G)$ and $E(\Delta)$ the topologies induced by the usual topologies of $C(G)$ and $C(\Delta)$ respectively.

If $G$ is not a Lie group, we have $E(G) \not\subset C(G)$ (cf. [4]).

For each $\alpha \in A$ let $\mathcal{D}(L_\alpha)$ denote the space of infinitely differentiable complex-valued functions defined on $L_\alpha$ having compact support, with the usual topology and $\mathcal{D}_\alpha(G) = \{f = f_\alpha \circ \phi_\alpha : f_\alpha \in \mathcal{D}(L_\alpha)\}$. We define $\psi_\alpha : \mathcal{D}(L_\alpha) \to \mathcal{D}_\alpha(G)$ by setting $\psi_\alpha(f_\alpha) = f_\alpha \circ \phi_\alpha$ and we take on $\mathcal{D}_\alpha(G)$ the topology transported by $\psi_\alpha$.

Following Bruhat, we define the space $\mathcal{D}(G)$ of infinitely differentiable complex-valued functions defined on $G$ having compact support, by setting $\mathcal{D}(G) = \bigcup_{\alpha \in A} \mathcal{D}_\alpha(G)$ and we equip $\mathcal{D}(G)$ with the inductive limit topology of the spaces $\mathcal{D}_\alpha(G)$. We can write $\mathcal{D}(G) = \lim_{\alpha \in A} \mathcal{D}(L_\alpha)$.

Let us consider the sequence of compacts $(\Delta_j)_{j \in \mathbb{N}}$ (cf. p. 2) and let $\mathcal{D}(\Delta_j)$ be the subspace of $\mathcal{D}(G)$ formed by the functions having their support contained in...
Δ_j. Let Δ_{j,α} = φ_α(Δ_j), ∀α ∈ A; we have \( D(Δ_j) = \bigcup_{α ∈ A} ψ_α(D(A_{j,α})) \) so we can write \( D(Δ_j) = \lim_{α \to A} D(Δ_{j,α}) \) and \( D(G) = \lim_{j \in N} D(Δ_j) \).

The partial-differentiation operators corresponding to a fundamental family \( S \) of one parameter subgroups of \( G \) are related with the partial-differentiation operators corresponding to the fundamental families \( S_α \) of one parameter subgroups of each \( L_α \) (cf. Th. 1) in the following way: \( D_x(f_α (φ_α)) = (D_{x_i,α} f_α) \circ φ_α \) if \( φ_α \circ x_i = x_{i,α} \).

We can easily show that those operators are continuous in \( D(G) \).

Let \( G = L, L \) a \( n \)-dimensional Lie group, and \( V_α \) an elementary neighborhood of \( a ∈ L \). We can define in \( C(V_α) \) an operator \( P_{x_i} f = \sum b_i f_\alpha (V_α) \). We easily verify that \( P_{x_i} P_{x_j} = P_{x_j} P_{x_i}, 1 ≤ i, j ≤ n \).

\[
P_{x_i}f(u) = \int_{b_i}^{t_i} f \left[u + \left(x_1(t_1) + \ldots + x_i(s_i) + \ldots + x_n(t_n)\right)\right] ds_i \bigg|_{t_1=\ldots=t_i=0} \quad \forall f ∈ C(V_α), \ ∀u ∈ V_α,
\]

with \((b_i) ∈ \mathbb{Q} = \xi_α(V_α)\). We easily verify that \( P_{x_i} P_{x_j} = P_{x_j} P_{x_i}, 1 ≤ i, j ≤ n \).

If \( p = (p_i) ∈ \mathbb{N}_0^n \), we set \( P^p = \prod_{i \in I} P^p_{x_i} \).

### 2 – Axiomatic definition of a space \( \mathbb{C}(G) \) of distributions on \( G \)

Let \( S = \{x_i : i ∈ I\} \) be a fundamental family of one parameter subgroups of \( G = \lim_{α \in A} L_α \) and \( Δ \) the closure of an open relatively compact subset of \( G \). We set \( M = \{p = (p_i) ∈ \mathbb{N}_0^n : p_i = 0 \) except for a finite number of indexes \( i ∈ I\}\).

The following axiomatic is formulated in terms of “continuous functions”, “addition” and “differentiation”, its logical universe being the space \( \mathbb{E}(Δ) \) that we characterize as follows:

**Axiom 1.** \( \mathbb{E}(Δ) ⊂ E(Δ) \).

**Axiom 2.** There exists an operation named “addition” that to every couple \( T_1, T_2 ∈ \mathbb{E}(Δ) \) associates an element \( T ∈ \mathbb{E}(Δ) \), named the addition of \( T_1 \) and \( T_2 \) and denoted by \( T_1 + T_2 \), such that if \( T_1 = f_1 ∈ E(Δ) \) and \( T_2 = f_2 ∈ E(Δ) \) then \( T_1 + T_2 \) is the addition in the usual sense.

**Axiom 3.** For each \( i ∈ I \) there exists an operator \( \tilde{D}_{x_i} : \mathbb{E}(Δ) → \mathbb{E}(Δ) \) such that:
1) If $T = f \in E(\Delta)$ and $D_{x_i} f \in E(\Delta)$ then $\tilde{D}_{x_i} T = D_{x_i} f$.
2) $\tilde{D}_{x_i}(T_1 + T_2) = \tilde{D}_{x_i} T_1 + \tilde{D}_{x_i} T_2$.
3) $\tilde{D}_{x_i} \tilde{D}_{x_j} T = \tilde{D}_{x_j} \tilde{D}_{x_i} T$, $\forall i, j \in I$.

We name the operator $\tilde{D}_{x_i}$ ($i \in I$) “generalized partial differentiation operator with respect to $x_i$.” If $p = (p_\ell) \in \mathbb{N}_0^I$ we set $\tilde{D}^p = \prod_{\ell \in I} \tilde{D}^{p_\ell}$.

**Axiom 4.** For each $T \in E(\Delta)$ there exists a finite number of continuous functions $f_k \in E(\Delta)$ and a finite number of multi-indexes $p_k \in M$ such that

$$T = \sum_{k=1}^{m} \tilde{D}^{p_k} f_k.$$ 

**Axiom 5.**

$$\sum_{k=1}^{m} \tilde{D}^{p_k} f_k = \sum_{j=1}^{s} \tilde{D}^{p_j} g_j$$

with $f_k = f_{\alpha_k} \circ \phi_{\alpha_k}$, $1 \leq k \leq m$, and $g_j = g_{\alpha_j} \circ \phi_{\alpha_j}$, $1 \leq j \leq s$, if and only if there exists $r \in M$, $r \geq p_1, \ldots, p_m, q_1, \ldots, q_s$ such that we have, for $\gamma > \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_s$, for each $a \in \Delta$ and for each elementary neighborhood $V_a$, of $a_\gamma = \phi_\gamma(a)$,

$$\sum_{k=1}^{m} p^{r-p_k} \left( (f_{\alpha_k} \circ \phi_{\alpha_k})_{V_a} \right) - \sum_{j=1}^{s} p^{r-q_j} \left( (g_{\beta_j} \circ \phi_{\beta_j})_{V_a} \right) = h_\gamma,$$

$h_\gamma$ being a pseudo-polynomial in $V_a$, with degree less then $r$ (cf. $n^0 1$).

To construct a model for our axiomatic, we consider the set $\mathcal{P}(M \times E(\Delta))$ of the finite parts of $M \times E(\Delta)$ and the equivalence relation $*$ defined as follows:

$$\left\{(p_1, f_{\alpha_1} \circ \phi_{\alpha_1}), \ldots, (p_m, f_{\alpha_m} \circ \phi_{\alpha_m})\right\} * \left\{(q_1, g_{\beta_1} \circ \phi_{\beta_1}), \ldots, (q_s, g_{\beta_s} \circ \phi_{\beta_s})\right\}$$

if and only if there exists $r \in M$, $r \geq p_1, \ldots, p_m, q_1, \ldots, q_s$ such that we have, for $\gamma > \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_s$, for each $a \in \Delta$ and each elementary neighborhood $V_a$, of $a_\gamma = \phi_\gamma(a)$,

$$\sum_{k=1}^{m} p^{r-p_k} \left( (f_{\alpha_k} \circ \phi_{\alpha_k})_{V_a} \right) - \sum_{j=1}^{s} p^{r-q_j} \left( (g_{\beta_j} \circ \phi_{\beta_j})_{V_a} \right) = h_\gamma,$$

$h_\gamma$ being a pseudo-polynomial in $V_a$, with degree less then $r$.

Denoting by $\left\{[(p_1, f_{\alpha_1}), \ldots, (p_m, f_{\alpha_m})] \right\}$ the equivalence class of $\left\{(p_1, f_{\alpha_1}), \ldots, (p_m, f_{\alpha_m})\right\}$ we set...
1) \[ [(p_1, f_1), \ldots, (p_m, f_m)] + [(q_1, g_1), \ldots, (q_s, g_s)] = \]
\[ = [(p_1, f_1), \ldots, (p_m, f_m)] \cup [(q_1, g_1), \ldots, (q_s, g_s)]; \]

2) \[ \lambda[(p_1, f_1), \ldots, (p_m, f_m)] = [(p_1, \lambda f_1), \ldots, (p_m, \lambda f_m)], \forall \lambda \in \mathbb{C}; \]

3) \[ \tilde{D}_{x_j}[(p_1, f_1), \ldots, (p_m, f_m)] = [(q_1, f_1), \ldots, (q_m, f_m)], \]
with \( p_k = (p_{k,i})_{i \in I}, 1 \leq k \leq m, \) and \( q_k = (q_{k,i})_{i \in I}, 1 \leq k \leq m, \) such that
\[ q_{k,i} = \begin{cases} p_{k,i} & \text{if } i \neq j, \\ p_{k,i} + 1 & \text{if } i = j. \end{cases} \]

Taking \( \tilde{E}(\Delta) = \mathcal{P}(M \times E(\Delta))/* \) with the operations defined above, we easily verify that \( \tilde{E}(\Delta) \) satisfies the precedent axioms.
For each multi-index \( p \in M \) we set
\[ \tilde{E}_p(\Delta) = \left\{ T \in \tilde{E}(\Delta) : T = \tilde{D}^p f \text{ with } f \in E(\Delta) \right\} \]
and
\[ N_p(\Delta) = \left\{ f \in E(\Delta) : \tilde{D}^p f = 0 \right\} \]
and define an isomorphism \( \tilde{D}^p f \to f + N_p(\Delta) \) between each \( \tilde{E}_p(\Delta) \) and each quotient vector space \( E(\Delta)/N_p(\Delta) \). We take on each \( \tilde{E}_p(\Delta) \) the quotient topology. The sets
\[ B_{p,n} = \left\{ T = \tilde{D}^p f : f \in E(\Delta) \text{ and } \|f\| < 1/n \right\}, \quad n \in \mathbb{N}, \]
form a basis of balanced neighborhoods of zero in \( \tilde{E}_p(\Delta) \).

We take on \( \tilde{E}(\Delta) \) the finest locally compact topology for which the natural injections of \( \tilde{E}_p(\Delta) \) into \( \tilde{E}(\Delta) \), \( p \in M \), are continuous.

The operators \( \tilde{D} \) are continuous on \( \tilde{E}(\Delta) \).

Let us consider the sequence of compacts \( (\Delta_j)_{j \in \mathbb{N}} \) (cf. p. 2). We can define homomorphisms \( \rho_{ij} : \tilde{E}(\Delta_j) \to \tilde{E}(\Delta_i) \), using the restriction mapping, that is
\[ \rho_{ij} \left( \sum_{k=1}^{m} \tilde{D}^{p_k} f_k \right) = \sum_{k=1}^{m} \tilde{D}^{p_k} (f_k|_{\Delta_i}), \quad i \leq j. \]

We obtain a projective system of topological spaces, \( (\tilde{E}(\Delta_j), \rho_{ij})_{i,j \in \mathbb{N}}, \) and we set \( \tilde{E}(G) = \lim_{j \in \mathbb{N}} \tilde{E}(\Delta_j) \). As \( E(G) \nsubseteq C(G) \), we call distribution on \( G \) to every element of the completion \( \tilde{C}(G) \) of \( \tilde{E}(G) \). Observe that if we denote by \( \tilde{C}(\Delta) \) the completion of each \( E(\Delta) \), we have \( \tilde{C}(G) = \lim_{j \in \mathbb{N}} \tilde{C}(\Delta_j) \). When \( G = \mathbb{R}^n \) the space \( \tilde{C}(G) \) coincides with the space of distributions constructed in [6].
3 - Relation between the spaces $\tilde{C}(G)$ and $\mathcal{D}'(G)$

The space $\mathcal{D}'(G)$ of the Bruhat distributions, being the strong dual of $\mathcal{D}(G)$, is the topological projective limit of the spaces $\mathcal{D}'(L_\alpha)$, $\mathcal{D}'(G) = \lim\limits_{\alpha \in A} \mathcal{D}'(L_\alpha)$ and the canonical maps are the transposed $^t\psi_\alpha$ and $^t\psi_{\alpha\beta}$ of $\psi_\alpha$ and $\psi_{\alpha\beta}$ respectively. We also have $\mathcal{D}'(\Delta_j) = \lim\limits_{\alpha \in A} \mathcal{D}'(\Delta_{j,\alpha})$ and $\mathcal{D}'(G) = \lim\limits_{j \in \mathbb{N}} \mathcal{D}'(\Delta_j)$.

Let $\Delta$ be the closure of an open relatively compact subset of $G$, $\mu$ the Haar measure of $G$, $\theta \in C(G)$ and $T \in \tilde{E}(\Delta)$. From Axiom 4 we have $T = \sum_{k=1}^{m} \tilde{D}^k f_k$ with $f_k \in E(\Delta)$. As $\mathcal{D}(G)$ is dense in $C_0(G)$ (cf. [1]), there exist sequences $(h_{k,r})_{r \in \mathbb{N}}$ in $\mathcal{D}(G)$, $1 \leq k \leq m$, such that $h_{k,r} \to f_k$, uniformly on $\Delta$. Then, the sequence $(h_r)_{r \in \mathbb{N}}$ in $\mathcal{D}(G)$, $h_r = \sum_{k=1}^{m} \tilde{D}^k h_{k,r}$ converges to $T$ in $\tilde{E}(\Delta)$.

We set, by definition:

$$\int_{\Delta} T(u) \theta(u) d\mu(u) = \lim_r \int_{\Delta} h_r(u) \theta(u) d\mu(u),$$

when this limit exists and is independent of the sequence $(h_r)_{r \in \mathbb{N}}$.

**Lemma.** Each space $\tilde{E}(\Delta_j)$ is isomorphic to a topological subspace of $\mathcal{D}'(\Delta_j)$.

**Proof:** If $G = \mathbb{R}^n$ or $G$ is a $n$-dimensional Lie group, the expression

$$T(\theta) = \int_{\Delta} T(u) \theta(u) d\mu(u)$$

defines a topological isomorphism $T \leftrightarrow T$ between $\tilde{E}(\Delta_j) = \tilde{C}(\Delta_j)$ and $\mathcal{D}'(\Delta_j)$. The first case was studied in [6]. For the second case, we can cover the compact $\Delta_j$ by a finite number of elementary neighborhoods and take a partition of unity in $\mathcal{D}(G)$ subordinated to this covering. It is then sufficient to use again [6] and the canonical atlas of $G$.

In the general case $G = \lim_{\alpha \in A} L_\alpha$, we take for each $\alpha \in A$ the Haar measure $\mu_{\alpha} = \phi_\alpha(\mu)$ of $L_\alpha$, and we define a topological isomorphism $H_{j,\alpha}$ from $\tilde{C}(\Delta_{j,\alpha})$ onto $\mathcal{D}'(\Delta_{j,\alpha})$, $T_\alpha \leftrightarrow T_\alpha$, by setting

$$T_\alpha(\theta_\alpha) = \int_{\Delta_{j,\alpha}} T_\alpha(u_\alpha) \theta_\alpha(u_\alpha) d\mu_\alpha(u_\alpha), \quad \forall \theta_\alpha \in \mathcal{D}(\Delta_{j,\alpha}).$$
Detailing, if \( T_\alpha = \sum_{k=1}^{m} \tilde{D}^{p_k} f_{\alpha,k} \), we have for each \( \theta_\alpha \in \mathcal{D}(\Delta_{j,\alpha}) \),

\[
(H_{j,\alpha}(T_\alpha))(\theta_\alpha) = T_\alpha(\theta_\alpha) = \sum_{k=1}^{m} \int_{\Delta_{j,\alpha}} (\tilde{D}^{p_k} f_{\alpha,k})(u_\alpha) \theta_\alpha(u_\alpha) \, d\mu_\alpha(u_\alpha) \\
= \sum_{k=1}^{m} (-1)^{|p_k|} \int_{\Delta_{j,\alpha}} f_{k,\alpha}(u_\alpha) (\tilde{D}^{p_k} \theta_\alpha)(u_\alpha) \, d\mu_\alpha(u_\alpha) .
\]

We consider continuous homomorphisms \( \tilde{\psi}_{\alpha \beta} : \tilde{C}(\Delta_{j,\alpha}) \to \tilde{C}(\Delta_{j,\beta}) \) defined by \( \tilde{\psi}_{\alpha \beta}(\tilde{D}^{p_{\alpha}} f_{\alpha}) = \tilde{D}^{p_{\beta}} (f_{\alpha} \circ \phi_{\alpha \beta}) \) with \( p_\alpha = (p_{\alpha,i}) \in \mathbb{N}_0^n \) and \( p_\beta = (p_{\beta,i}) \in \mathbb{N}_0^n \) such that

\[
p_{\beta,i} = \begin{cases} 
  p_{\alpha,i} & \text{if } \phi_{\alpha \beta} \circ x_{i,\beta} = x_{i,\alpha} \in S_\alpha, \\
  0 & \text{if } \phi_{\alpha \beta} \circ x_{i,\beta} = 0 , 
\end{cases}
\]

We also consider the continuous homomorphisms \( \tilde{\psi}_{\alpha} : \tilde{C}(\Delta_{j,\alpha}) \to \tilde{E}(\Delta_j) \) defined by \( \tilde{\psi}_{\alpha}(\tilde{D}^{p_{\alpha}} f_{\alpha}) = \tilde{D}^{p}(f_{\alpha} \circ \phi_{\alpha}) \) with \( p = (p_i) \in M \) such that

\[
p_i = \begin{cases} 
  p_{\alpha,i} & \text{if } \phi_{\alpha} \circ x_i = x_{i,\alpha} \in S_\alpha, \\
  0 & \text{if } \phi_{\alpha} \circ x_i = 0 , 
\end{cases}
\]

and we obtain

\[
\tilde{E}(\Delta_j) = \bigcup_{\alpha \in A} \tilde{\psi}_{\alpha}(\tilde{C}(\Delta_{j,\alpha})) .
\]

If \( \alpha < \beta \)

\[
(H_{j,\beta}(\tilde{D}^{p_{\beta}} (f_{\alpha} \circ \phi_{\alpha \beta}))))(\theta_\alpha \circ \phi_{\alpha \beta}) = \\
= \int_{\Delta_{j,\beta}} (\tilde{D}^{p_{\beta}} (f_{\alpha} \circ \phi_{\alpha \beta}))(u_{\beta}) \theta_\beta(u_{\beta}) \, d\mu_\beta(u_{\beta}) \\
= \int_{\Delta_{j,\alpha}} (\tilde{D}^{p_{\alpha}} f_{\alpha})(u_\alpha) \theta_\alpha(u_\alpha) \, d\mu_\alpha(u_\alpha) \\
= (H_{j,\alpha}(\tilde{D}^{p_{\alpha}} f_{\alpha}))(\theta_\alpha) ,
\]

and so

\[
(H_{j,\beta}(\tilde{\psi}_{\alpha \beta}(T_\alpha)))(\theta_\alpha \circ \phi_{\alpha \beta}) = (H_{j,\alpha}(T_\alpha))(\theta_\alpha), \quad \forall T_\alpha \in \tilde{C}(\Delta_{j,\alpha}), \quad \forall \theta_\alpha \in \mathcal{D}(\Delta_{j,\alpha}) .
\]

Then we can consider a homomorphism \( H_j : \tilde{E}(\Delta_j) \to \mathcal{D}'(\Delta_j) \) defined by

\[
H_j(\tilde{\psi}_{\alpha}(T_\alpha)) = (H_{j,\beta}(\tilde{\psi}_{\alpha \beta}(T_\alpha)))_{\beta > \alpha} .
\]

As each \( H_{j,\alpha} \) is one-to-one, \( H_j \) is also one-to-one. We easily verify that

\[
(H_j(T))(\theta) = (H_{j,\alpha}(\sum_{k=1}^{m} \tilde{D}^{p_k} f_k))(\theta) = \int_{\Delta_j} T(u) \, \theta(u) \, d\mu(u) = \\
= \sum_{k=1}^{m} (-1)^{|p_k|} \int_{\Delta_j} f_k(u) \, (\tilde{D}^{p_k} \theta)(u) \, d\mu(u), \quad \forall T \in \tilde{E}(\Delta_{j,\alpha}), \quad \forall \theta \in \mathcal{D}(\Delta_{j,\alpha}) .
\]
To prove that $H_j$ is continuous it is sufficient to verify that if $(T_r)_{r \in \mathbb{N}}$ is a sequence in $E_p(\Delta_j)$ such that $T_r \to 0$ there, that is $T_r = \tilde{D}^p f_r$ and $f_r \to 0$ uniformly on $\Delta_j$, then $H_j(T_r) \to 0$ on every bounded subset $B$ of $D(\Delta_j)$. From [1] there exists $\alpha \in A$ such that $B = \psi_\alpha(B_\alpha)$ with $B_\alpha$ bounded subset of $D(\Delta_{j,\alpha})$. Then there exists $b \in \mathbb{R}^+$ such that

$$
\sup_{u \in \Delta_j} |(D^p \theta)(u)| = \sup_{u_\alpha \in \Delta_{j,\alpha}} |(\tilde{D}^p \theta_\alpha)(u_\alpha)| < b, \quad \forall \theta \in B
$$

and

$$
(H_j(T_r))(\theta) = \left| \int_{\Delta_j} (\tilde{D}^p f_r)(u) \theta(u) \, d\mu(u) \right|
$$

$$
= \left| (-1)^{|p|} \int_{\Delta_j} f_r(u) D^p \theta(u) \, d\mu(u) \right|
$$

$$
\leq \sup_{u \in \Delta_j} |f_r(u)| b \mu(\Delta_j), \quad \forall \theta \in B,
$$

so $H_j(T_r) \to 0$ in $D'(\Delta_j)$.

Let us prove that $H_j$ is bicontinuous.

We consider a basis of neighborhoods of zero in $\tilde{E}_p(\Delta_j)$, $p \in M$,

$$
B_{p,n} = \left\{ T = \tilde{D}^p f : f \in \tilde{E}(\Delta_j) \text{ and } \|f\| < 1/n \right\}, \quad n \in \mathbb{N}.
$$

The balanced convex hulls of the sets $\bigcup_{p \in M} B_{p,n}$, $n \in \mathbb{N}$, form a basis of neighborhoods of zero in $\tilde{E}(\Delta_j)$, $\{B_n : n \in \mathbb{N}\}$. We consider in each $\tilde{C}(\Delta_{j,\alpha})$ the corresponding basis of neighborhoods $\{B_{n,\alpha} : n \in \mathbb{N}\}$: we have $B_n = \bigcup_{\alpha \in A} \psi_\alpha(B_{n,\alpha})$.

As each $H_{j,\alpha}$ is bicontinuous, $W_{n,\alpha} = H_{j,\alpha}(B_{n,\alpha})$ is a neighborhood of zero in $D'(\Delta_j)$. Let us take $W_n = (\psi^{-1}_{\alpha}(W_{n,\alpha}) \cap H_j(\tilde{E}(\Delta_j)))$: $W_n$ is a neighborhood of zero in $H_j(\tilde{E}(\Delta_j))$ for the topology induced by $D'(\Delta_j)$. We must prove that $H_{j,\alpha}^{-1}(W_n) \subset B_{n,\alpha}$.

Noting that

$$
(\psi^{-1}_\alpha((H_j \circ \tilde{\psi}_\alpha)(T_\alpha)))(\theta_\alpha) = ((H_j \circ \tilde{\psi}_\alpha)(T_\alpha))(\theta_\alpha \circ \phi_\alpha)
$$

$$
= \int_{\Delta_j} (\tilde{\psi}_\alpha(T_\alpha))(u) ((\theta_\alpha \circ \phi_\alpha)(u) \, d\mu(u)
$$

$$
= \int_{\Delta_{j,\alpha}} T_\alpha(u_\alpha) \theta_\alpha(u_\alpha) \, d\mu_\alpha(u_\alpha) = (H_{j,\alpha}(T_\alpha))(\theta_\alpha),
$$

\forall T_\alpha \in \tilde{C}(\Delta_{j,\alpha}), \forall \theta_\alpha \in D(\Delta_{j,\alpha}),
we have \( t \psi_\alpha \circ H_j \circ \tilde{\psi}_\alpha = H_{j,\alpha}, \forall \alpha \in A, \) so

\[
B_{n,\alpha} = H_{j,\alpha}^{-1}(W_{n,\alpha}) = \tilde{\psi}_{\alpha}^{-1}\left[H_j^{-1}\left(t \psi_\alpha^{-1}(W_{n,\alpha}) \cap H_j(\tilde{E}(\Delta_j))\right) \cap \tilde{\psi}_\alpha(\tilde{C}(\Delta_{j,\alpha}))\right]
\]

and

\[
\tilde{\psi}_\alpha(B_{n,\alpha}) = H_j^{-1}\left[t \psi_\alpha^{-1}(W_{n,\alpha}) \cap H_j(\tilde{E}(\Delta_j))\right] \cap \tilde{\psi}_\alpha(\tilde{C}(\Delta_{j,\alpha})).
\]

Finally,

\[
H_j^{-1}(W_n) = \bigcup_{\alpha \in A} \left[H_j^{-1}(W_n) \cap \tilde{\psi}_\alpha(\tilde{C}(\Delta_{j,\alpha}))\right]
\]

\[
= \bigcup_{\alpha \in A} \left[H_j^{-1}(t \psi_\alpha^{-1}(W_{n,\alpha}) \cap H_j(\tilde{E}(\Delta_j))) \cap \tilde{\psi}_\alpha(\tilde{C}(\Delta_{j,\alpha}))\right]
\]

\[
= \bigcup_{\alpha \in A} \tilde{\psi}_\alpha(B_{n,\alpha}) = B_n.
\]

**Corollary.** The space \( \tilde{E}(G) \) is isomorphical to a topological subspace of \( \mathcal{D}'(G) \).

**Theorem 2.** The space \( \tilde{C}(G) \) is isomorphical to the space \( \mathcal{D}'(G) \).

**Proof:** As \( \mathcal{D}(G) \) is dense in \( \mathcal{D}'(G) \) (cf. [1]) and \( \mathcal{D}(G) \subset \tilde{E}(G) \), \( \tilde{E}(G) \) is also dense in \( \mathcal{D}'(G) \). As \( \mathcal{D}'(G) \) induces on \( \tilde{E}(G) \) its own topology, the completion \( \tilde{C}(G) \) of \( \tilde{E}(G) \) is isomorphical to \( \mathcal{D}'(G) \).

**Final Remark.** As J.S. Silva refers, namely in the introduction of his paper “Integrals and orders of growth of distributions” (Proceedings of an International Summer Institute held in Lisbon, September 1964), we believe that the characterization of the distributions \( T \) on \( G \) in terms of “continuous functions”, “addition” and “differentiation” can afford the introduction of the Fourier transformation for distributions by means of the integral

\[
\int_G T(u) \chi(-u) \, d\mu(u), \quad \chi \in \hat{G},
\]

without assuming any previous theory of the same transformation for functions.

We hope to publish soon a paper about the Fourier transformation theory for distributions on parametrizable groups.
BIBLIOGRAPHY


Suzana Metello de Nápoles,
C.M.A.F., Av. Prof. Gama Pinto, 2,
1699 Lisboa Codex – PORTUGAL