AN APPROXIMATION PROCEDURE FOR FIXED POINTS OF STRONGLY LIPSCHITZ OPERATORS

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Abstract: Based on a modified iterative algorithm, fixed points of the operators of the form $S = T + U$ on nonempty closed convex subsets of Hilbert spaces are approximated. Here $T$ is strongly Lipschitz and Lipschitz continuous and $U$ is Lipschitz continuous.

1 – Introduction

Recently, Wittmann [5, Theorem 2] approximated fixed points of nonexpansive mappings $T$ on nonempty closed convex subsets of Hilbert spaces by employing an iterative procedure

$$x_n = (1 - a_n) x_0 + a_n T x_{n-1} \quad \text{for } n \geq 1,$$

where $\{a_n\}$ is an increasing sequence in $[0, 1)$ such that

$$\lim_{n \to \infty} a_n = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} (1 - a_n) = \infty.$$

This result, for example, applies to $a_n = 1 - n^{-a}$ with $0 < a \leq 1$, and improves a theorem of Halpern [1, Theorem 3] which does not apply to the case $a_n = 1 - 1/n$. Furthermore, (2) is not just sufficient, but also necessary for the convergence of $\{x_n\}$ for all $T$ [1, Theorem 2].

Here we are concerned with the approximation of fixed points of operators of the form $S = T + U$, where $T$ is strongly Lipschitz and Lipschitz continuous and $U$ is Lipschitz continuous on a nonempty closed convex subset $K$ of a real Hilbert
space $H$ by using the following modified iterative procedure in a more general setting

$$x_{n+1} = (1 - a_n) x_n + a_n \left[ (1 - t) x_n + t (T + U) x_n \right] \text{ for } n \geq 0,$$

where $t > 0$ is arbitrary and the sequence $\{x_n\}$ lies in $[0, 1]$ such that $\sum_{n=0}^{\infty} a_n$ diverges for all $n \geq 0$.

For $U = 0$ in (3), we find the iterative algorithm

$$x_{n+1} = (1 - a_n) x_n + a_n \left[ (1 - t) x_n + t T x_n \right] \text{ for all } n \geq 0.$$

For $t = 1$ in (4), we arrive at

$$x_{n+1} = (1 - a_n) x_n + a_n T x_n \text{ for all } n \geq 0.$$

### 2 – Preliminaries

Let $H$ be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

**Definition 2.1** An operator $T: H \to H$ is said to be strongly Lipschitz if, for all $u, v$ in $H$, there exists a real number $r \geq 0$ such that

$$\langle Tu - Tv, u - v \rangle \leq -r \|u - v\|^2.$$

The operator $T$ is called Lipschitz continuous if there exists a real number $s > 0$ such that

$$\|Tu - Tv\| \leq s \|u - v\| \text{ for all } u, v \text{ in } H.$$

It is easily seen that (7) implies that

$$\langle Tu - Tv, u - v \rangle \leq s \|u - v\|^2 \text{ for all } u, v \text{ in } H.$$

**Definition 2.2.** An operator $T: H \to H$ is said to be hemicontinuous if, for all $u, v$ in $H$, the function

$$t \to \langle T(tu + (1-t)v), u - v \rangle \text{ for } 0 \leq t \leq 1,$$

is continuous.
To this end, let us consider an example of strongly Lipschitz operator where the real number \( r \) in inequality (6) is slightly relaxed.

**Example 2.1** [6]: Let \( K \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( T : K \to K \) be hemicontinuous on \( K \) such that, for a real number \( r > -1 \) and for all \( u, v \) in \( K \),

\[
\langle Tu - Tv, u - v \rangle \leq -r\|u - v\|^2.
\]

Then \( T \) has a unique fixed point in \( K \).

3 – The fixed point theorem

In this section we consider the approximation of fixed points of a combination of strongly Lipschitz and Lipschitz continuous operators.

**Theorem 3.1.** Let \( H \) be a real Hilbert space and let \( K \) be a non-empty closed convex subset of \( H \). Let \( T : K \to K \) be strongly Lipschitz and Lipschitz continuous with respective real numbers \( r \geq 0 \) and \( s \geq 1 \), and let \( U : K \to K \) be Lipschitz continuous with a real number \( m > 0 \). Let \( F \) be a nonempty set of fixed points of \( S = T + U \), and let \( \{a_n\} \) be a sequence in \([0,1]\) such that \( \sum_{n=0}^{\infty} a_n \) diverges for all \( n \geq 0 \). Then, for any \( x_0 \) in \( K \), the sequence \( \{x_n\} \) generated by the iterative algorithm (3) for

\[
0 \leq k = \left[ \left( (1-t)^2 - 2t(1-t) r + t^2 s^2 \right)^{1/2} + t m \right] < 1
\]

for all \( t \) such that \( 0 < t < 2(1+r-m)/(1+2r+s^2-m^2) \) and \( 1 + r - m > 0 \), converges to a fixed point of \( S = T + U \).

When \( U = 0 \) in Theorem 3.1, we arrive at the following result.

**Corollary 3.1.** Let \( K \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( T : K \to K \) be strongly Lipschitz and Lipschitz continuous with corresponding constants \( r \geq 0 \) and \( s \geq 1 \). Let \( \{a_n\} \) be a sequence in \([0,1]\) such that \( \sum_{n=0}^{\infty} a_n \) diverges for all \( n \geq 0 \). Then, for any element \( x_0 \) in \( K \), the sequence \( \{x_n\} \) generated by the iterative algorithm (4) for

\[
0 \leq k_0 = \left[ \left( (1-t)^2 - 2t(1-t) r + t^2 s^2 \right)^{1/2} \right] < 1
\]

for all \( t \) such that \( 0 < t < 2(1+r)/(1+2r+s^2) \), converges to a fixed point of \( T \).
Proof of Theorem 3.1: For an element \( z \) in \( F \), we have

\[
\|x_{n+1} - z\| = \left\|(1 - a_n) x_n + a_n \left[ (1 - t) x_n + t(T + U) x_n \right] - (1 - a_n) z \right\|
\]

\[
= \left\| a_n \left[ (1 - t) z + t(T + U) z \right] \right\|
\]

\[
= \left\| (1 - a_n) (x_n - z) + a_n \left[ (1 - t) (x_n - z) + t(Tx_n - Tz) + t(Ux_n - Uz) \right] \right\|
\]

\[
\leq (1 - a_n) \|x_n - z\| + a_n \|(1 - t) (x_n - z) + t(Tx_n - Tz)\| + a_n t \|Ux_n - Uz\|
\]

Since \( T \) is strongly Lipschitz and Lipschitz continuous, this implies that

\[
\left\| (1 - t) (x_n - z) + t(Tx_n - Tz) \right\|^2 = (1 - t)^2 \|x_n - z\|^2 + 2t(1 - t) \left\langle Tx_n - Tz, x_n - z \right\rangle + t^2 \|Tx_n - Tz\|^2
\]

\[
\leq \left[ (1 - t)^2 - 2t(1 - t) r + t^2 s^2 \right] \|x_n - z\|^2
\]

Applying (14) to (13) and using the Lipschitz continuity of \( U \), it follows that

\[
\|x_{n+1} - z\| \leq \left\{ (1 - a_n) \left[ \left( (1 - t)^2 - 2t(1 - t) r + t^2 s^2 \right)^{1/2} + t m \right] a_n \right\} \|x_n - z\|
\]

\[
= \left[ 1 - (1 - k) a_n \right] \|x_n - z\|
\]

\[
\leq \prod_{j=0}^{n} \left[ 1 - (1 - k) a_j \right] \|x_0 - z\|
\]

where \( 0 \leq k = [(1 - t)^2 - 2t(1 - t)r + t^2 s^2)^{1/2} + tm] < 1 \) for all \( t \) such that \( 0 < t < 2(1 + r - m)/(1 + 2r + s^2 - m^2) \) for \( 1 + r - m > 0 \). Since \( \sum_{j=0}^{\infty} a_j \) diverges and \( k < 1 \), this implies that \( \lim_{n \to \infty} \prod_{j=0}^{n} [1 - (1 - k) a_j] = 0 \), and consequently, \( \{ x_n \} \) converges to \( z \), a fixed point of \( S = T + U \). This completes the proof.

REFERENCES


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