LINEAR FLOW IN POROUS MEDIA
WITH DOUBLE PERIODICITY

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Abstract: We study the classical steady Stokes equations with homogeneous Dirichlet boundary conditions. We work in a 3-D domain which contains solid obstacles, two-periodically distributed with period $\varepsilon$ (respectively $\varepsilon^2$), where $\varepsilon$ is a small parameter. Our aim is to study the asymptotic behaviour, as $\varepsilon \to 0$. We use the 3-scale convergence for getting the 3-scale limit problem. The problem obtained is a three-pressures system.

Résumé: On étudie le problème de Stokes stationnaire classique, avec des conditions de Dirichlet homogènes au bord. Le problème est posé dans un domaine qui contient des inclusions solides réparties périodiquement, avec périodicité de l’ordre d’un petit paramètre $\varepsilon$ et de l’ordre de $\varepsilon^2$. Pour le passage à la limite en $\varepsilon$, on utilise la méthode de convergence 3-échelle. Le problème limite 3-échelle obtenu est un problème à trois pressions.

Introduction

We study here the homogenization of the Stokes steady flow in double periodic media. We will apply the multi-scale convergence method, introduced by G. Allaire, M. Briane [1]. This method generalizes the two-scale convergence method introduced by G. Nguetseng [10] for the simply periodic domains.

The problem presented here was first treated by J.-L. Lions [8]. The method used for getting the limit problem was the formal expansion of the velocity and of the pressure. The results we present here justify the expansions.

In §1 we give the mathematical model of the problem. We define the domain which has two parts: the fluid part and the solid part. The solid part is made

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by solid obstacles two-periodically distributed, with period $\varepsilon$ (respectively $\varepsilon^2$), where $\varepsilon$ is a small parameter.

In §2 we give a priori estimates and convergence results for the velocity. Next we recall and prove some results related on the three-scale convergence.

In §3 we construct the extension of the pressure to the whole of $\Omega$ and we give a convergence result. The difficulty is here the construction of an extension operator for the pressure to the solid part of the domain. We already know some methods for constructing such an extension (cf. L. Tartar [12], R. Lipton, M. Avellaneda [9], C. Conca [2], I.-A. Ene, J. Saint Jean Paulin [5]). The last two methods are applied for a problem with Neumann type boundary conditions at the fluid-solid interface. The extension presented here is a generalization of the method presented in L. Tartar [12].

In §4 we pass to the limit as $\varepsilon \to 0$ in the initial problem. We obtain the 3-scale limit system, which represents a three-pressures problem.

The Stokes problem in double periodic media was already studied by T. Lévy [7] and P. Donato, J. Saint Jean Paulin [3], but the domain presented here is a different one. The solid obstacles periodically distributed with period $\varepsilon$ in the domain presented here are replaced in [7] and [3] by the fluid, which corresponds to a porous fissured rock.

An analogous result for the Poisson equation in porous fissured rocks was studied by P. Donato, J. Saint Jean Paulin [4].

1 - Positionning of the problem

Let $\Omega$ be a bounded open domain of boundary $\partial \Omega$ in $\mathbb{R}^N, N \geq 2$.

Let us consider two sets $Y = \prod_{i=1}^{N} [0,1[$ and $Z = \prod_{i=1}^{N} [0,1[$ and two closed subsets $Y_s \subset Y$, $Z_s \subset Z$, with non-empty interior, contained in $Y$ (respectively $Z$). We define:

$$Y^* = Y \setminus Y_s, \quad Z^* = Z \setminus Z_s.$$ 

Let $\varepsilon$ be a small positive parameter. Let us suppose that there exists an $\varepsilon$ such that the domain $\overline{Y}$ is exactly covered by a finite number of cells $\varepsilon \overline{Z}$. Moreover, let us suppose that $Y_s$ is exactly covered by a finite number of cells $\varepsilon Z$. This last hypothesis implies some restrictions for the geometry of $Y_s$ (see an example in Figure 1.1). We deduce that there is no intersection between the solid obstacles $Y_s$ and $\varepsilon Z_s$ in the cell $Y$, as we can see in Figure 1.3. If we consider all the small parameters $\frac{\varepsilon}{2^n}$, the above assumptions are still true.
We multiply the new cell (Figure 1.3) by $\varepsilon$ and we repeat it in the domain $\Omega$. We assume (for simplicity), that $\overline{\Omega}$ is exactly covered by a finite number of cells $\varepsilon Y$'. We define $\Omega_\varepsilon$ by taking out of $\Omega$ the domains $\varepsilon Y_s$ and $\varepsilon^2 Z_s$. Let us notice that there is no intersection between the solid obstacles $\varepsilon Y_s$ and $\varepsilon^2 Z_s$ in $\Omega_\varepsilon$, because there is no intersection between the solid obstacles $Y_s$ and $Z_s$ in the cell $Y$. The domain $\Omega_\varepsilon$ (which corresponds to the fluid) is connected, but the union of solid obstacles is not connected.

Let $\chi_{Y^*}$ and $\chi_{Z^*}$ be the characteristic functions of the domains $Y^*$ and $Z^*$, defined by:

$$
\chi_{Y^*}(y) = \begin{cases} 
1 & \text{in } Y^*, \\
0 & \text{in } Y \setminus Y^* 
\end{cases}, \quad
\chi_{Z^*}(z) = \begin{cases} 
1 & \text{in } Z^*, \\
0 & \text{in } Z \setminus Z^* 
\end{cases}.
$$

We extend the characteristic functions $\chi_{Y^*}$ (respectively $\chi_{Z^*}$) by periodicity, with period 1 in $y_i$ and in $z_i$, for $i = 1, ..., N$. The domain $\Omega_\varepsilon$, defined as above is described by:

$$
\Omega_\varepsilon = \left\{ x \mid x \in \Omega, \chi_{Y^*}\left(\frac{x}{\varepsilon}\right) \chi_{Z^*}\left(\frac{x}{\varepsilon^2}\right) = 1 \right\}.
$$
The domain $\Omega_{\varepsilon}$ presents a double periodicity, with small solid obstacles of order $\varepsilon$ and with very small obstacles of order $\varepsilon^2$. This domain modelizes a rigid porous medium with double periodicity.

We define the boundary of $\Omega_{\varepsilon}$, denoted by $\partial\Omega_{\varepsilon}$ and composed by three parts:

- the boundary of obstacles $\varepsilon Y_s$,
- the boundary of obstacles $\varepsilon^2 Z_s$,
- the boundary of $\Omega$.

In $\Omega_{\varepsilon}$ defined as above, we consider the following Stokes problem:

\[
\begin{cases}
-\varepsilon^2 \Delta u^\varepsilon + \nabla p^\varepsilon = f \quad \text{in } \Omega_{\varepsilon}, \\
\text{div } u^\varepsilon = 0 \quad \text{in } \Omega_{\varepsilon}, \\
u^\varepsilon = 0 \quad \text{on } \partial\Omega_{\varepsilon}.
\end{cases}
\]

The first relation in (1.2) represents the classical steady Stokes equation. The term $\varepsilon^2$ represents the order of fluid's viscosity. This assumption is not essential for a linear problem, because we can always rescale. The second relation is the incompressibility condition of the fluid. On the boundary of $\Omega_{\varepsilon}$ we consider Dirichlet homogeneous conditions. We recall that for a domain $D$ we define the spaces $L^2(D)$ and $H_0^1(D)$ by:

\[
\begin{align*}
L^2(D) &= (L^2(D))^N, \\
H_0^1(D) &= \left\{ \psi \in L^2(D), \frac{\partial \psi}{\partial x_i} \in L^2(D), \psi = 0 \text{ on } \partial D \right\}.
\end{align*}
\]
The exterior body forces are denoted by \( f \). The function \( f = (f_i)_{i=1,\ldots,N} \) belongs to \( \mathbb{L}^2(\Omega) \) and the right hand side of relation (1.2) represents the restriction of \( f \) to \( \Omega_{\varepsilon} \). The existence and the uniqueness of a solution \((u^\varepsilon, p^\varepsilon) \in \mathbb{H}^1_0(\Omega_{\varepsilon}) \times \mathbb{L}^2(\Omega) / \mathbb{R} \) for (1.2) is classical (see R. Temam [13]).

2 – A priori estimates and convergence results for the velocity

Our aim is to study the asymptotic behaviour, as \( \varepsilon \to 0 \), of the solution of problem (1.2). For passing to the limit we need extensions of velocity and pressure to the whole of \( \Omega \). We first give a Poincaré’s type lemma, adapted at the domain presented in §1:

**Lemma 2.1.** For any function \( \phi \in \mathbb{H}^1_0(\Omega_{\varepsilon}) \), we have:

\[
|\phi|_{\mathbb{L}^2(\Omega_{\varepsilon})} \leq c \varepsilon^2 |\nabla \phi|_{\mathbb{L}^2(\Omega_{\varepsilon})}^N .
\]

In the following, \( c \) denotes a constant independent of \( \varepsilon \).

Let \( \tilde{u}^\varepsilon \) be the extension of \( u^\varepsilon \) by zero to the whole of \( \Omega \). For the function \( \tilde{u}^\varepsilon \) we can easily prove the following a priori estimates:

**Proposition 2.2.** If \( u^\varepsilon \) is solution of (1.2), then we have:

\[
|\nabla \tilde{u}^\varepsilon|_{\mathbb{L}^2(\Omega_{\varepsilon})}^N \leq c ,
\]

\[
|\tilde{u}^\varepsilon|_{\mathbb{L}^2(\Omega_{\varepsilon})} \leq c \varepsilon^2 .
\]

Before establishing convergence of the velocity, we first recall and prove some general results adapted below to our case.

Let us denote by \( C^\infty_p(Y \times Z) \) the space of \( C^\infty \) functions, \( Y \)-periodic and \( Z \)-periodic. We have the following lemma:

**Lemma 2.3** (G. Allaire, M. Briane [1]). Let \( v^\varepsilon \) be a sequence of bounded functions in \( \mathbb{L}^2(\Omega) \). Then there exists a subsequence still denoted \( v^\varepsilon \) and a function \( v \in \mathbb{L}^2(\Omega \times Y \times Z) \) such that:

\[
\lim_{\varepsilon \to 0} \int_{\Omega} v^\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) dx = \iint_{\Omega Y Z} v(x, y, z) \varphi(x, y, z) dx dy dz ,
\]
for every function \( \varphi(x, y, z) \in L^2(\Omega, C^\infty_p(Y \times Z)) \). We say that \( v^\varepsilon \) 3-scale converges to \( v \). Moreover,

\[
v^\varepsilon \rightharpoonup v_0 = \int_Y \int_Z v \, dy \, dz \quad \text{weakly in} \quad L^2(\Omega).
\]

As in G. Allaire, M. Briane [1], we prove the result:

**Proposition 2.4.** Let \( v^\varepsilon \) be a bounded sequence in \( L^2(\Omega) \) which 3-scale converges to \( v \) and such that

\[
(2.5)\quad \text{div} \, v^\varepsilon = 0 \quad \text{in} \quad \Omega.
\]

Then the limit \( v \) satisfies the following relations:

\[
(2.6)\quad \text{div}_x \int_Y \int_Z v \, dz \, dy = 0,
\]

\[
(2.7)\quad \text{div}_y \int_Z v \, dz = 0,
\]

\[
(2.8)\quad \text{div}_z v = 0.
\]

**Proof:** Let \( \varphi \) be a function in \( \mathcal{D}(\Omega) \). We multiply relation (2.5) by \( \varphi \) and integrating by parts, we get:

\[
0 = \lim_{\varepsilon \to 0} \int_{\Omega} (\text{div} \, v^\varepsilon(x)) \varphi(x) \, dx = - \lim_{\varepsilon \to 0} \int_{\Omega} v^\varepsilon \nabla \varphi \, dx.
\]

But

\[
\lim_{\varepsilon \to 0} \int_{\Omega} v^\varepsilon \nabla \varphi \, dx = \int_{\Omega} \int_Y \int_Z v(x, y, z) \nabla \varphi(x) \, dx \, dy \, dz
\]

since \( v^\varepsilon \) 3-scale converges to \( v \).

We deduce

\[
\int_{\Omega} \text{div}_x \left( \int_Y \int_Z v \, dz \, dy \right) \varphi(x) \, dx = 0, \quad \forall \varphi \in \mathcal{D}(\Omega),
\]

which implies (2.6). Multiplying (2.5) by particular functions \( \varphi \in \mathcal{D}(\Omega, C^\infty_p(Y)) \) and \( \varphi \in \mathcal{D}(\Omega, C^\infty_p(Y \times Z)) \), we obtain the relations (2.7)--(2.8).

**Remark 2.5.** For a set \( D \), let \( \mathbb{H}^1_p(D) \) be the space of functions \( \mathbb{H}^1_{\text{loc}}(\mathbb{R}^N) \) which are \( D \)-periodic.
Choosing particular test functions $\varphi \in H^1(\Omega)$ (respectively $\varphi \in H^1(Z)$) in relation (2.7) (respectively (2.8)), we obtain the following periodicity condition:

$$
\left[ \int_Z v(x, y, z) \, dz \right] \nu_Y \text{ takes opposite values on opposite faces of } Y,
$$

$$
v \nu_Z \text{ takes opposite values on opposite faces of } Z,
$$

where $\nu_Y$ (resp. $\nu_Z$) represents the unit outward normal to $Y$ (resp. $Z$).

For the velocity’s extension $\tilde{v}$ we prove the following results:

**Proposition 2.6.** Let $\tilde{v}$ be defined as before. Then there exists $u \in L^2(\Omega Y Z)$ such that, up to a subsequence, we have:

(2.9) $\quad \varepsilon^{-2} \tilde{v} \to u \quad 3$-scale,

(2.10) $\quad u(x, y, z) = 0 \quad \text{in } \Omega Y_s Z_s$,

(2.11) $\quad \varepsilon^{-2} \tilde{v} \to u_0 = \int_{Y^* Z^*} u \, dz \, dy \quad \text{weakly in } L^2(\Omega),

(2.12) $\quad \nabla \tilde{v} \to \nabla u \quad 3$-scale.

**Proof:** Relation (2.9) is a direct consequence of Lemma 2.3 applied for $v = \varepsilon^{-2} \tilde{v}$. This is possible according to estimate (2.3).

For proving (2.10) we note that, for $v^\varepsilon = \varepsilon^{-2} \tilde{v}$, relation (2.4) becomes:

$$
\lim_{\varepsilon \to 0} \int_{\Omega} \varepsilon^{-2} v^\varepsilon(x) \varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}) \, dx = \int_{\Omega Y Z} u(x, y, z) \varphi(x, y, z) \, dz \, dy \, dx.
$$

We choose a test function $\varphi$ such that $\varphi = 0$ in $\Omega \times Y^* \times Z^*$. Using $\tilde{v} = 0$ in $\Omega \setminus \Omega_\varepsilon$, we deduce:

$$
0 = \int_{\Omega Y Z} u(x, y, z) \varphi(x, y, z) \, dz \, dy \, dx,
$$

which implies (2.10).

Relation (2.11) is a direct consequence of Lemma 2.3 and of relation (2.10). For proving relation (2.12) we note that relation (2.2) and Lemma 2.3 imply the existence of a function $\xi \in [L^2(\Omega Y Z)]^N$ such that:

$$
\lim_{\varepsilon \to 0} \int_{\Omega} \nabla \tilde{v}^\varepsilon \varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}) \, dx = \int_{\Omega Y Z} \xi(x, y, z) \varphi(x, y, z) \, dz \, dy \, dx.
$$
Integrating the left hand term by parts, we get:

\[
\int_\Omega \nabla u^\varepsilon \varphi \left( x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) dx = - \int_\Omega \tilde{u} \text{div} \varphi \left( x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) dx \\
= - \int_\Omega \tilde{u} \left( \text{div}_x \varphi + \frac{1}{\varepsilon} \text{div}_y \varphi + \frac{1}{\varepsilon^2} \text{div}_z \varphi \right) dx \\
= - \int_\Omega \varepsilon^{-2} \left( \varepsilon^2 \tilde{u} \text{div}_x \varphi + \tilde{u} \text{div}_y \varphi + \tilde{u} \text{div}_z \varphi \right) dx .
\]

Passing to the limit in \( \varepsilon, \) we derive:

\[
- \int_\Omega \int_Y \int_Z u(x,y,z) \text{div}_z \varphi(x,y,z) \, dz \, dy \, dx = \int_\Omega \int_Y \int_Z \xi(x,y,z) \varphi(x,y,z) \, dz \, dy \, dx .
\]

Integrating the left hand side of the previous relation by parts we deduce:

\[
\int_\Omega \int_Y \int_Z \left[ \xi(x,y,z) - \nabla_z u(x,y,z) \right] \varphi(x,y,z) \, dz \, dy \, dx = 0 , \quad \forall \varphi \in \mathcal{D}(\Omega, C^\infty_p(Y \times Z)) ,
\]

consequently \( \xi(x,y,z) = \nabla_z u(x,y,z) , \) which ends the proof. \( \blacksquare \)

**Proposition 2.7.** Let \( u \) be the function defined by Proposition 2.6. Then we have:

\[
(2.13) \quad \text{div}_x \int_{Y^* Z^*} u \, dz \, dy = 0 ,
\]

\[
(2.14) \quad \text{div}_y \int_{Z^*} u \, dz = 0 ,
\]

\[
(2.15) \quad \text{div}_z u = 0 ,
\]

\[
(2.16) \quad \left[ \int_{Y^* Z^*} u \, dy \, dz \right] \cdot \nu = 0 \quad \text{on} \quad \partial \Omega ,
\]

\[
(2.17) \quad u \cdot \nu_Z \text{ takes opposite values on opposite faces of } Z ,
\]

\[
(2.18) \quad \left[ \int_{Z^*} u \, dz \right] \cdot \nu_Y \text{ takes opposite values on opposite faces of } Y ,
\]

\[
(2.19) \quad u \cdot \nu_Z = 0 \quad \text{on} \quad \partial Z_s ,
\]

\[
(2.20) \quad \left[ \int_{Z^*} u \, dz \right] \cdot \nu_Y = 0 \quad \text{on} \quad \partial Y_s .
\]
Proof: The relations (2.13)–(2.15) are a consequence of Proposition 2.4 applied to the sequence \( \varepsilon^{-2} \tilde{w} \) (which 3-scale converges to \( u \)) and of relation (2.10).

In order to obtain relation (2.16), we use the linearity and the continuity of the normal trace application from

\[
\mathbb{H}(\text{div}, \Omega) = \left\{ \psi \in L^2(\Omega) \mid \text{div} \psi \in L^2(\Omega) \right\}
\]

into \( \mathbb{H}^{-\frac{1}{2}}(\partial \Omega) \), then we use the relations (2.10)–(2.11).

The relations (2.17)–(2.18) are a consequence of Remark 2.5 and of relation (2.10).

Multiplying (2.15) by a test function \( \psi \in \mathbb{H}^1_p(Z) \) and using (2.17) we obtain relation (2.19). We get relation (2.20) by multiplying (2.14) by \( \psi \in \mathbb{H}^1_p(Y) \) and using (2.18).

3 – Extension of the pressure and convergence result

We now construct a restriction operator \( S_{\varepsilon} \) from \( \mathbb{H}^1(\Omega) \) into \( \mathbb{H}^1(\Omega_\varepsilon) \). Using this operator we will define an extension for the pressure to the whole of \( \Omega \).

Let \( Y_{\varepsilon} \) be the domain defined by:

\[
Y_{\varepsilon} = Y \setminus \left( Y_\varepsilon \cup \bigcup \varepsilon Z_\varepsilon \right).
\]

We define the space \( \mathbb{H}^1_s(Y_{\varepsilon}) \) by:

\[
\mathbb{H}^1_s(Y_{\varepsilon}) = \left\{ \phi \in \mathbb{H}^1(\Omega) \mid \phi = 0 \text{ on } \partial Y_\varepsilon \text{ and on } \partial \left( \bigcup \varepsilon Z_\varepsilon \right) \right\}.
\]

We define the space \( \mathbb{H}^1_s(Y^*) \) by:

\[
\mathbb{H}^1_s(Y^*) = \left\{ \phi \in \mathbb{H}^1(Y^*) \mid \phi = 0 \text{ on } \partial Y_\varepsilon \right\}.
\]

To prove the claimed result, we first construct a restriction operator \( R \) from the space \( \mathbb{H}^1(Y) \) into the space \( \mathbb{H}^1_s(Y^*) \) and next we construct the operator \( W_\varepsilon \) from the space \( \mathbb{H}^1_s(Y^*) \) into the space \( \mathbb{H}^1_s(Y_{\varepsilon}) \). Using operators \( R \) and \( W_\varepsilon \), we construct the operator

\[
S_\varepsilon : \mathbb{H}^1(Y) \to \mathbb{H}^1_s(Y_{\varepsilon})
\]

and next we define \( S_{\varepsilon} \) by applying \( S_\varepsilon \) to each period \( \varepsilon Y \) of \( \Omega \).
So we construct \( S_{e2} \) in three steps, corresponding to the three following lemmas.

**Lemma 3.1.** There exists a restriction operator

\[
R : H^1(Y) \to H^1_s(Y^*)
\]

such that for \( v \in H^1(Y) \) we have:

\[
\begin{align*}
(3.1) & \quad Rv = v \quad \text{if} \quad v = 0 \quad \text{in} \quad Y_s, \\
(3.2) & \quad \text{div} \, Rv = 0 \quad \text{in} \quad Y^* \quad \text{if} \quad \text{div} \, v = 0 \quad \text{in} \quad Y, \\
(3.3) & \quad |Rv|_{H^1(Y)} \leq c |v|_{H^1(Y)}. 
\end{align*}
\]

**Proof:** Let us consider a smooth surface \( \gamma \) strictly contained in \( Y \), enclosing \( Y_s \). We denote by \( \widetilde{Y}_s \) the domain between \( \gamma \) and \( \partial Y_s \).

![Figure 3.1 - Domain Y.](image)

As in Lemma 3 of L. Tartar [12], we have the following result:

If \( v \in H^1(Y) \), there exist \( w \in H^1(\widetilde{Y}_s) \), \( q \in L^2(\widetilde{Y}_s)/\mathbb{R} \) such that:

\[
\begin{cases}
-\Delta w = -\Delta v + \nabla q & \text{in } \widetilde{Y}_s, \\
\text{div} \, w = \text{div} \, v + \frac{1}{|Y_s|} \int_{Y_s} \text{div} \, v \, dy & \text{in } \widetilde{Y}_s, \\
w|_{\gamma} = v|_{\gamma}, & w|_{\partial Y_s} = 0,
\end{cases}
\]

where \( |\widetilde{Y}_s| \) represents the measure of \( Y_s \). Moreover, there exists a constant \( c \) independent of \( v \) such that:

\[
|w|_{H^1(\widetilde{Y}_s)} \leq c |v|_{H^1(Y)}.
\]
Let us notice that:

\[ Y = Y^* \cup Y_s = (Y^* \setminus \bar{Y}_s) \cup \bar{Y}_s \cup Y_s. \]

We define the operator \( R \) by:

\[
Rv(y) = \begin{cases} 
  v(y) & \text{if } y \in Y^* \setminus \bar{Y}_s, \\
  w(y) & \text{if } y \in \bar{Y}_s, \\
  0 & \text{if } y \in Y_s.
\end{cases}
\]

This definition and properties satisfied by the function \( v \) imply the relations (3.1)–(3.3).

**Lemma 3.2.** There exists a restriction operator

\[ W_\varepsilon : H^1_s(Y^*) \to H^1_s(Y_f) \]

such that for \( Rv \in H^1_s(Y^*) \) we have:

\[
W_\varepsilon(Rv) = Rv \quad \text{if } Rv = 0 \quad \text{in } \bigcup(\varepsilon Z_s),
\]

\[
\text{div} W_\varepsilon(Rv) = 0 \quad \text{in } Y_f \quad \text{if } \text{div} Rv = 0 \quad \text{in } Y^*,
\]

\[
\varepsilon^2 |\nabla W_\varepsilon(Rv)|^2_{L^2(Y_f)^N} + |W_\varepsilon(Rv)|^2_{L^2(Y_f)} \leq c |v|^2_{H^1(Y)}.
\]

**Proof:** Let \( H^1_s(Z^*) \) be the space defined by:

\[ H^1_s(Z^*) = \left\{ \phi \in H^1(Z^*) \mid \phi = 0 \quad \text{in } \partial Z_s \right\}. \]

In the fixed cell \( Z \), let us consider a smooth surface \( \tilde{\gamma} \) strictly contained in \( Z \). We denote by \( \tilde{Z}_s \) the domain between \( \tilde{\gamma} \) and \( \partial Z_s \). The domain \( \tilde{Z}_s \) is independent of the parameter \( \varepsilon \).

![Fig. 3.2 – Domain Z.](image)
As in lemma 3 of L. Tartar [12], we have the following result. If \( u \in H^1(Z) \), there exist \( w \in H^1(\tilde{Z}_s) \), \( q \in L^2(\tilde{Z}_s) / \mathbb{R} \) such that:

\[
\begin{aligned}
-\Delta \bar{w} &= -\Delta \bar{u} + \nabla \bar{q} & \text{in } \tilde{Z}_s, \\
\text{div } \bar{w} &= \text{div } \bar{u} + \frac{1}{|Z_s|} \int_{\gamma_s} \text{div } \bar{u} \, dy & \text{in } \tilde{Z}_s, \\
\bar{w}|_{\gamma} &= \bar{u}|_{\gamma}, & \bar{w}|_{\partial Z_s} = 0.
\end{aligned}
\]

Moreover, there exists a constant \( c \) independent of \( \bar{u} \) such that:

\[ |\bar{w}|_{H^1(\tilde{Z}_s)} \leq c |\bar{u}|_{H^1(Z)}. \]

Let us notice that \( Z = Z^* \cup Z_s = (Z^* \setminus \tilde{Z}_s) \cup \tilde{Z}_s \cup Z_s \).

For every function \( \bar{u} \in H^1(Z) \) we construct an application

\[ W : H^1(Z) \rightarrow H^1_s(Z^*) \]

defined by:

\[
W(\bar{u})(z) = \begin{cases} 
\bar{u}(z) & \text{if } z \in Z^* \setminus \tilde{Z}_s, \\
\bar{u}(z) & \text{if } z \in \tilde{Z}_s, \\
0 & \text{if } z \in Z_s,
\end{cases}
\]

(3.7)

and satisfying:

\[ W(\bar{u}) = \bar{u} \quad \text{if } \bar{u} = 0 \text{ in } Z_s, \]
\[ \text{div } W(\bar{u}) = 0 \quad \text{if } \text{div } \bar{u} = 0, \]
\[ |W(\bar{u})|_{H^1(Z)} \leq c |\bar{u}|_{H^1(Z)}. \]

(3.8)

Fig. 3.3 – Cell \( Y \) with solid obstacle \( Y_s \) and obstacles \( \varepsilon Z_s \).
Next we apply $W$ to every period $\varepsilon Z$ of $Y \setminus Y_s$ and we obtain a function $W_\varepsilon$,

$$W_\varepsilon : \mathbb{H}_s^1(Y^s) \to \mathbb{H}_s^1(Y_f^s),$$

satisfying the relations (3.4) and (3.5). We apply $W_\varepsilon$ to $Rv \in \mathbb{H}_s^1(Y^s)$ and using relation (3.8) we get:

$$\varepsilon^2 |\nabla W_\varepsilon(Rv)|^2_{L^2(Y_f^s)} + |W_\varepsilon(Rv)|^2_{L^2(Y_f^s)} \leq \varepsilon^2 |\nabla Rv|^2_{L^2(Y^s)} + |Rv|^2_{L^2(Y^s)}.$$

Since $|Rv|_{\mathbb{H}^1(Y)} \leq c |v|_{\mathbb{H}^1(Y)}$, we deduce:

$$\varepsilon^2 |\nabla W(Rv)|^2_{L^2(Y_f^s)} + |W(Rv)|^2_{L^2(Y_f^s)} \leq c |v|^2_{\mathbb{H}^1(Y)},$$

which is exactly (3.6).

**Lemma 3.3.** There exists a restriction operator

$$S_\varepsilon : \mathbb{H}_0^1(\Omega) \to \mathbb{H}_0^1(\Omega_\varepsilon)$$

such that:

\begin{align*}
(3.9) & \quad S_\varepsilon(v) = v \quad \text{in} \quad \Omega_\varepsilon, \; \forall v \in \mathbb{H}_0^1(\Omega_\varepsilon), \\
(3.10) & \quad \div S_\varepsilon v = 0 \quad \text{in} \quad \Omega_\varepsilon \quad \text{if} \quad \div v = 0 \quad \text{in} \quad \Omega, \\
(3.11) & \quad |\nabla S_\varepsilon v|^2_{L^2(\Omega_\varepsilon)} \leq c \left( \frac{1}{\varepsilon^2} |v|_{L^2(\Omega)} + \frac{1}{\varepsilon} |\nabla v|_{L^2(\Omega)} \right)^2, \\
(3.12) & \quad |S_\varepsilon v|_{L^2(\Omega_\varepsilon)} \leq c \left( |v|_{L^2(\Omega)} + |\nabla v|_{L^2(\Omega)} \right).
\end{align*}

**Proof:** Let $S_\varepsilon : \mathbb{H}^1(Y) \to \mathbb{H}^1(Y_f^s)$ be the application defined by:

$$S_\varepsilon v(y) = \begin{cases} 
W_\varepsilon(Rv)(y) & \text{if } y \in Y^s, \\
0 & \text{if } y \in Y_s.
\end{cases}$$

Using the construction of $W_\varepsilon$ we also have:

$$S_\varepsilon v(y) = \begin{cases} 
W_\varepsilon(Rv)(y) & \text{if } y \in Y_f^s, \\
0 & \text{otherwise},
\end{cases}$$

and $S_\varepsilon$ satisfies (3.9)–(3.10).

Due to (3.6), the application $S_\varepsilon$ satisfies:

$$\varepsilon^2 |\nabla S_\varepsilon v|^2_{L^2(Y_f^s)} + |S_\varepsilon v|^2_{L^2(Y_f^s)} \leq c |v|^2_{\mathbb{H}^1(Y)}.$$
We define $S_\varepsilon^2$ by applying $S_\varepsilon$ to each period $\varepsilon Y$. The relation (3.13) then implies:

$$
\varepsilon^4 |\nabla S_\varepsilon^2 v|^2_{L^2(\Omega_\varepsilon)} + |S_\varepsilon^2 v|^2_{L^2(\Omega_\varepsilon)} \leq c \left( |v|^2_{L^2(\Omega)} + \varepsilon^2 |\nabla v|^2_{L^2(\Omega)} \right),
$$

and we deduce relations (3.11)–(3.12).

Let $v$ be a function of $H^1_0(\Omega)$. As $\nabla p^\varepsilon \in H^{-1}(\Omega_\varepsilon)$, we define the application $F^\varepsilon$ by:

$$(F^\varepsilon, v)_{\Omega_\varepsilon} = \langle \nabla p^\varepsilon, S_\varepsilon^2 v \rangle_{\Omega_\varepsilon},$$

where $S_\varepsilon$ is the operator defined by Lemma 3.3. The following proposition gives us the extension of the pressure $p^\varepsilon$ to the whole $\Omega$. Moreover, we establish a strong convergence result for this extension. Following the ideas of L. Tartar [12], we can prove:

**Proposition 3.4.** Let $p^\varepsilon$ be as in (1.2). Then, for each $\varepsilon$ there exists an extension $P^\varepsilon$ of $p^\varepsilon$ defined on $\Omega$ such that:

$$P^\varepsilon = p^\varepsilon \quad \text{in} \quad \Omega_\varepsilon.$$

Moreover, up to a subsequence, we have:

$$(3.15) \quad P^\varepsilon \rightarrow p_0 \quad \text{strongly in} \quad L^2(\Omega)/\mathbb{R}.$$

The function $F^\varepsilon$ and the pressure $p^\varepsilon$ are linked by:

$$(3.16) \quad F^\varepsilon = \nabla P^\varepsilon.$$

**4 – Passage to the limit and 3-scale limit problem**

We recall that as in I.-A. Ene [6] we have the following “de Rham”-type result:

**Lemma 4.1.** Let $w \in L^2(\Omega \times Y \times Z)$ be a function satisfying:

$$(4.1) \quad \iint_{\Omega Y Z} w(x, y, z) \phi(x, y, z) \, dx \, dy \, dz = 0,$$

for all function $\phi$ belonging to $D(\Omega, C_0^\infty(Y \times Z))$ such that:

$$(4.2) \quad \text{div}_y \phi(x, y, z) = 0, \quad \text{div}_z \phi(x, y, z) = 0.$$
Then there exist two functions \( q_1 \in L^2(\Omega, H^1_p(Y)/\mathbb{R}) \) and \( q_2 \in L^2(\Omega \times Y, H^1_p(Z)/\mathbb{R}) \) such that:

\[
(4.3) \quad w(x, y, z) = \nabla_y q_1(x, y) + \nabla_z q_2(x, y, z).
\]

Let us recall that we denoted by \( u \) the 3-scale limit of \( \varepsilon^{-2} \widetilde{u}^\varepsilon \) (see relation (2.9)) and that \( p_0 \) (defined in relation (3.15)) represents the strong limit of the pressure’s extension in \( L^2(\Omega) \). Using Lemma 4.1 and 3-scale convergence results of \( \Omega_2 \Omega_3 \), we prove the main result of this paper:

**Theorem 4.2.** Let \( u \) and \( p_0 \) be as before. Then there exist \( p_1 \in L^2(\Omega, H^1_p(Y^*)/\mathbb{R}) \) and \( p_2 \in L^2(\Omega \times Y, H^1_p(Z^*)/\mathbb{R}) \) such that:

\[
(4.4) \quad -\Delta_z u + \nabla_x p_0 + \nabla_y p_1 + \nabla_z p_2 = f \quad \text{in} \quad \Omega \times Y^* \times Z^*.
\]

**Proof:** We recall the first equation of (1.2):

\[
-\varepsilon^2 \Delta u^\varepsilon + \nabla p^\varepsilon = f \quad \text{in} \quad \Omega^\varepsilon.
\]

We multiply it by a function \( \varphi \in D(\Omega, C^\infty_p(Y \times Z)) \) such that \( \text{div}_y \varphi(x, y, z) = 0 \) and \( \text{div}_z \varphi(x, y, z) = 0 \). Integrating the first term of the left hand side by parts we get:

\[
\varepsilon^2 \int_{\Omega^\varepsilon} \nabla u^\varepsilon(x) \nabla \varphi \left( x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) dx + \int_{\Omega^\varepsilon} \nabla p^\varepsilon(x) \varphi \left( x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) dx =
\]

\[
= \int_{\Omega^\varepsilon} f(x) \varphi \left( x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) dx.
\]

Using the definition of the extension \( \widetilde{u}^\varepsilon \), relations (3.14) and (3.16) and making the additional assumption \( \varphi(x, y, z) = 0 \) in \( \Omega \times Y_s \times Z_s \) (i.e. \( \varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}) \in H^1_0(\Omega_s) \)) we obtain:

\[
\varepsilon^2 \int_{\Omega} \nabla \widetilde{u}^\varepsilon(x) \nabla \varphi \left( x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) dx - \int_{\Omega} P^\varepsilon(x) \text{div}_x \varphi \left( x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) dx =
\]

\[
= \int_{\Omega} f(x) \varphi \left( x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) dx.
\]

Passage to the 3-scale limit in (4.5) implies:

\[
\int_{\Omega Y^* Z^*} \int \nabla_z u(x, y, z) \nabla_z \varphi(x, y, z) \, dx \, dy \, dz - \int_{\Omega Y^* Z^*} \int p_0(x) \text{div}_x \varphi(x, y, z) \, dx \, dy \, dz =
\]

\[
= \int_{\Omega Y^* Z^*} \int f(x) \varphi(x, y, z) \, dx \, dy \, dz.
\]
Hence,
\[
\int_{\Omega} \int_{Y^*} \int_{Z^*} \left[ -\Delta_z u(x, y, z) + \nabla_x p_0(x) - f(x) \right] \varphi(x, y, z) \, dx \, dy \, dz = 0.
\]

Using the particular form of \( \varphi \), Lemma 4.1 then implies relation (4.4).

Conclusions

Theorem 4.2 and the results of §2 imply the following three-scale system (4.6).

\[
\begin{align*}
-\Delta_z u + \nabla_x p_0 + \nabla_y p_1 + \nabla_z p_2 &= f & \text{in } \Omega \times Y^* \times Z^*, \\
\text{div}_x \int_{Y^*} \int_{Z^*} u \, dz \, dy &= 0 & \text{in } \Omega, \\
\text{div}_y \int_{Z^*} u \, dz &= 0 & \text{in } \Omega \times Y^*, \\
\text{div}_z u &= 0 & \text{in } \Omega \times Y^* \times Z^*, \\
\int_{Y^*} \int_{Z^*} u \, dy \, dz \cdot \nu &= 0 & \text{on } \partial\Omega, \\
u \cdot \nu_Z \text{ takes opposite values on opposite faces of } Z, \\
\int_{Z^*} u \, dz \cdot \nu_Y \text{ takes opposite values on opposite faces of } Y, \\
u \cdot \nu_Z &= 0 & \text{on } \partial Z_x, \\
\int_{Z^*} u \, dz \cdot \nu_Y &= 0 & \text{on } \partial Y_x.
\end{align*}
\]

Remark 4.3. System (4.6) is obtained in J.-L. Lions [8, Chapter 2, Section 3], with the method of asymptotic expansion on the velocity and of the pressure. The first equation is a three-pressure equation. The three pressures \( p_0, p_1, p_2 \) are the three first terms in the asymptotic expansion of the pressure \( p^e \). We recall here that, as in J.-L. Lions [8], we may write the function \( u \) in two different ways:

(i) The function \( u \) satisfies the homogenized equation (a Darcy-law type):
\[
u(x, y, z) = \phi(y, z) \left( f(x) - \nabla_x p_0(x) \right),
\]
where the function \( \phi \) is solution of a local problem in \( Y^* \times Z^* \).
With the notation \( M(\mathbf{y}) = 1 \int \int \phi(y, z) \, dy \, dz \), the function \( p_0 \) is solution of the following Neumann problem:

\[
(M(\mathbf{y}) (f - \nabla_x p_0), \nabla q) = 0, \quad \forall q \in H^1(\Omega).
\]

(ii) We can also express \( u \) by a relation depending on both pressures \( p_0 \) and \( p_1 \). We have:

\[
u(x, y, z) = \phi_1(z) \left( f - \nabla_x p_0(x) - \nabla_y p_1(x, y) \right),
\]

where \( \phi_1 \) is solution of a local problem in \( Z^\ast \) and the pressure \( p_1(x, y) \) is solution of the following Neumann problem:

\[
\left( M(\phi_1) \left( f - \nabla_x p_0(x) - \nabla_y p_1(x, y) \right), \nabla_y q_1 \right)_{Y^\ast} = 0, \\
\forall q_1 \in H^1(Y^\ast), \quad q_1 \text{ \( Y \)-periodic}.
\]

**Remark 4.4.** The results presented here may be generalized. Let \( r_\varepsilon \) be a parameter depending on \( \varepsilon \) such that:

\[
\frac{r_\varepsilon}{\varepsilon} \to 0 \quad \text{if} \quad \varepsilon \to 0.
\]

In the domain \( \Omega \) we replace the very small obstacles of order \( \varepsilon^2 \) by obstacles of order \( r_\varepsilon \), periodically distributed with periodicity \( r_\varepsilon \). We consider the problem:

\[
\begin{aligned}
- r_\varepsilon \Delta u_\varepsilon + \nabla p_\varepsilon &= f \quad \text{in} \quad \Omega_\varepsilon, \\
\text{div} \, u_\varepsilon &= 0 \quad \text{in} \quad \Omega_\varepsilon, \\
u_\varepsilon &= 0 \quad \text{on} \quad \partial \Omega_\varepsilon.
\end{aligned}
\]

The case already treated corresponds to \( r_\varepsilon = \varepsilon^2 \).

For the extension of the velocity, solution of (4.7), we can prove the convergences:

\[
r_\varepsilon^{-1} u_\varepsilon \to u \quad 3\text{-scale},
\]

\[
r_\varepsilon^{-1} u_\varepsilon \to u_0 = \int \int u \, dz \, dy \quad \text{weakly in} \quad L^2(\Omega),
\]

\[
\nabla u_\varepsilon \to \nabla_z u \quad 3\text{-scale}.
\]

For the strong convergence of pressure’s extension we have Proposition 3.4, which still holds. We can prove that velocity and pressure limits satisfy the system (4.6).
REFERENCES


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