ON NONHOMOGENEOUS BIHARMONIC EQUATIONS INVOLVING CRITICAL SOBOLEV EXPONENT

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Abstract: In this paper we consider the problem \( \Delta^2 u = \lambda |u|^{q_c-2} u + f \) in \( \Omega \), \( u = \Delta u = 0 \) on \( \partial \Omega \), where \( q_c = 2N/(N-4) \), \( N > 4 \), is the limiting Sobolev exponent and \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \). Under some restrictions on \( f \) and \( \lambda \), the existence of weak solution \( u \) is proved. Moreover \( u \geq 0 \) for \( f \geq 0 \) whenever \( \lambda \geq 0 \).

1 – Introduction

In this article, we show that the problem

\[
(P_{\lambda,f}) \quad \begin{cases} \Delta(\Delta u) = \lambda |u|^{q_c-2} u + f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial \Omega, \end{cases}
\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \), \( N > 4 \), \( \Delta \) is the Laplacian operator and \( q_c = 2N/(N-4) \), has weak solutions in \( H^2_0(\Omega) = H^2(\Omega) \cap H^1_0(\Omega) \) equipped with the norm

\[
\|u\|_{H^2_\theta} = \left( \int_\Omega |\Delta u|^2 \right)^{1/2}. 
\]

To this end we consider the functional

\[
F_\lambda(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 \, dx - \frac{\lambda}{q_c} \int_\Omega |u|^{q_c} \, dx - \int_\Omega f u \, dx, \quad u \in H^2_\theta(\Omega), \quad \lambda > 0. 
\]

Under some suitable conditions, it is proved that (1.1) admits at least two solutions. Our arguments make use of the mountain pass theorem and of the Lions concentration-compactness principle.
Recently, Van der Vorst [10] considered the following problem
\begin{equation}
S = \inf \left\{ \int_{\Omega} |\Delta u|^2; \; u \in H_0^2(\Omega), \; \int_{\Omega} |u|^{q_c} = 1 \right\}.
\end{equation}
He proved that the infimum in (1.3) is never achieved by a function \( u \in H_0^2(\Omega) \) when \( \Omega \) is bounded. In contrast Hadiji, Picard and the author in [7] considered the problem
\begin{equation}
S_\varphi = \inf \left\{ \int_{\Omega} |\Delta u|^2; \; u \in H_0^2(\Omega), \; \int_{\Omega} |u + \varphi|^{q_c} = 1 \right\}.
\end{equation}
They showed that the infimum in (1.4) is achieved whenever \( \varphi \) is continuous and non identically equal to zero. More precisely it is shown that, for any minimizing sequence \((u_m)\) for (1.4), there exists a subsequence \((u_{m_k})\) and a function \( u \in H_0^2(\Omega) \) such that
\[
\begin{align*}
\lim_{k \to \infty} u_{m_k} & \rightharpoonup u \text{ weakly in } H_0^2(\Omega) \quad \text{and} \quad \|u + \varphi\|_{q_c} = 1.
\end{align*}
\]

On the other hand, Bernis et al. [1] considered a variant of (1.1) where \( f \) is replaced by \( \beta |u|^{p-2} u, \; 1 < p < 2 \). They proved the existence of at least two positive solutions for \( \beta \) sufficiently small. At this stage, we would like to mention that when \( \Omega = \mathbb{R}^N \) P.L. Lions [9] proved that \( S \) is achieved only by the function \( u_\varepsilon \) defined by
\[
\begin{align*}
u_\varepsilon(x) = \frac{
[(N - 4) (N - 2) N (N + 2) \varepsilon^2
\begin{align*}
&\left\{ \Delta (\Delta u) = |u|^{q_c-2} u + g \quad \text{in} \; \Omega, \\
&u = \Delta u = 0 \quad \text{on} \; \partial \Omega,
\end{align*}
\end{align*}
\right.
\right. \varepsilon + |x - a|^2)^{\frac{N+4}{2}}}, \quad x \in \mathbb{R}^N,
\]
for any \( a \in \mathbb{R}^N \) and any \( \varepsilon > 0 \). This note is organized as follows. In Section 2 we verify that \( F_\lambda \) satisfies the (PS)_c condition. In Section 3 we prove the existence of a local minimizer \( u \) of \( F_\lambda \). Moreover, we show that \( u \geq 0 \) whenever \( f \geq 0 \) and \( \lambda \geq 0 \). Section 4 is devoted to the existence of a second solution to (1.4). The results presented in this paper have been announced in [6].

Notice that if \( f \equiv 0 \), the result of Section 3 is valid and gives the trivial solution \( u = 0 \). The method we adopt is closely related to the one of [3].

Before the verification of the (PS)_c condition, let us remark that if \( v \) is a solution to (1.1) then \( u = \lambda^{\frac{1}{p-2}} v \) satisfies
\begin{equation}
\begin{align*}
\left\{ \Delta (\Delta u) = |u|^{q_c-2} u + g \quad \text{in} \; \Omega, \\
&u = \Delta u = 0 \quad \text{on} \; \partial \Omega,
\end{align*}
\end{equation}
where \( g = \lambda^{\frac{1}{p-2}} f \).
2 – Verification of the \((PS)_c\) condition

Let \(\Omega\) be a bounded domain in \(\mathbb{R}^N\), \(N > 4\), and \(f \in L^2(\Omega)\). We denote by 
\(F_\lambda : H^2_\theta(\Omega) \to \mathbb{R}\) the functional defined by

\[
F_\lambda(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 \, dx - \frac{\lambda}{q_c} \int_\Omega |u|^{q_c} \, dx - \int_\Omega fu \, dx ,
\]

(2.1)

where \(\Delta\) is the Laplacian operator and \(\lambda\) is a real parameter. We first look for

critical points of \(F \overset{\text{def}}{=} F_1\). We show that \(F\) satisfies the Palais–Smale condition

in a suitable sublevel strip. 

Let \(S\) be the best Sobolev embedding constant of \(H^2_\theta(\Omega)\) into \(L^{q_c}(\Omega)\); that is

\[
S = \inf \left\{ \int_\Omega |\Delta u|^2 ; \; u \in H^2_\theta(\Omega), \; \int_\Omega |u|^{q_c} = 1 \right\}
\]

(2.2)

and

\[
K = \frac{N \frac{q}{q_c}}{2 q (4 q_c)^{\frac{q}{q_c}}} \|f\|_q , \quad q = \frac{q_c}{q_c - 1} .
\]

(2.3)

**Proposition 2.1.** The functional \(F\) satisfies the \((PS)_c\) condition in the

sublevel strip \((-\infty, \frac{2}{N} S^{N^2} - K)\); that is if \(\{u_m\}\) is a sequence in \(H^2_\theta(\Omega)\) such that

\[
F(u_m) \to c \quad \text{and} \quad dF(u_m) \to 0 \quad \text{in} \; H^{-2}_\theta(\Omega) ,
\]

(2.4)

where

\[
c < \frac{2}{N} S^{N^2} - K ,
\]

then \(\{u_m\}\) contains a subsequence which converges strongly in \(H^2_\theta(\Omega)\).

**Proof:** Let \(\{u_m\}\) be a sequence in \(H^2_\theta(\Omega)\) which satisfies (2.4). From (2.4) it

is easy to see that \(\{u_m\}\) is bounded in \(H^2_\theta(\Omega)\); thus there is a subsequence \(\{u_{m_k}\}\),

and an element \(u\) of \(H^2_\theta(\Omega)\) such that

\[
u_{m_k} \rightharpoonup u \quad \text{weakly in} \; H^2_\theta(\Omega)
\]

(2.5)

and

\[
u_{m_k} \to u \quad \text{strongly in} \; L^p(\Omega) , \quad 1 \leq p < q_c \quad \text{and a.e. in} \; \overline{\Omega} .
\]

(2.6)

The concentration-compactness Lemma of Lions [9] asserts the existence of at

most a countable index set \(J\) and positive constants \(\{\nu_j\}, \; j \in J\) such that

\[
|\nu_{m_k}|^{q_c} \to |u|^{q_c} + \sum_{j \in J} \nu_j \delta_{x_j} ,
\]

(2.7)
weakly in the sense of measures, and

\[ |\Delta u_{m_k}|^2 \to \mu, \]  

for some positive bounded measure \( \mu \). Moreover,

\[ \mu \geq |\Delta u|^2 + \sum_{j \in J} S \nu_j^{\frac{N-4}{2}} \delta_{x_j}, \]  

where

\[ x_j \in \Omega \quad \text{and} \quad \nu_j = 0 \quad \text{or} \quad \nu_j \geq S^{\frac{N}{2}}. \]  

We assert that \( \nu_j = 0 \) for each \( j \). If not, assume that \( \nu_{j_0} \neq 0 \), for some \( j_0 \). From the hypothesis (2.4),

\[ c = \lim_{k \to \infty} F(u_{m_k}) - \frac{1}{2} \left\langle \frac{dF(u_{m_k})}{du_{m_k}}, u_{m_k} \right\rangle, \]

\[ c \geq \frac{2}{N} \int_{\Omega} |u|^q - \frac{1}{2} \int_{\Omega} f u + \frac{2}{N} S^{\frac{N}{2}}. \]

Using the Hölder inequality one has

\[ c \geq \frac{2}{N} S^{\frac{N}{2}} - \frac{N}{2q (1+q_c)^{\frac{2}{q}}} \|f\|_q^q. \]

This contradicts the hypothesis. Consequently \( \nu_j = 0 \) for each \( j \) and

\[ \lim_{k \to \infty} \int_{\Omega} |u_{m_k}|^q = \int_{\Omega} |u|^q, \]

which implies

\[ u_{m_k} \to u \quad \text{strongly in} \quad H^{q}_{0}(\Omega). \]

The proof is complete.

3 – Existence of a solution

In this part we consider the problem of finding solutions to \((P_{\lambda, f})\). We show, under suitable conditions on \( f \) and \( \lambda \), that \( F_{\lambda} \) has an infimum on a small ball in \( H^{q}_{0}(\Omega) \). We suppose first that \( \lambda = 1 \), and denote by \( F \) the functional \( F_1 \). The proof is based on the following lemma.
Lemma 3.1. There exist constants $r$ and $R > 0$ such that if $\|f\|_2 \leq R$, then
\begin{equation}
F(u) \geq 0 \quad \text{for all } \|u\|_{H^2_0(\Omega)} = r.
\end{equation}

Proof: Thanks to the Sobolev and Hölder inequalities we have
\begin{equation}
F(u) \geq \frac{1}{2} \int_\Omega |\Delta u|^2 - \frac{1}{q_c} S^{-q_c} \left( \int_\Omega |\Delta u|^2 \right)^{\frac{q_c}{2}} - |\Omega|^{\frac{1}{2} - \frac{1}{q_c}} S^{-1} \|f\|_2 \left( \int_\Omega |\Delta u|^2 \right)^{1/2}.
\end{equation}

Inequality (3.2) can be written
\begin{equation}
F(u) \geq h \left( \|u\|_{H^2_0} \right),
\end{equation}
where
\[ h(x) = \frac{1}{2} x^2 - \lambda_0 x^{q_c} - \lambda_1 x, \quad \lambda_0 = \frac{1}{q_c} S^{-q_c} \quad \text{and} \quad \lambda_1 = \|f\|_2 |\Omega|^{\frac{1}{2} - \frac{1}{q_c}} S^{-1}. \]

Let
\[ g(x) = \frac{1}{2} x - \lambda_0 x^{q_c-1} - \lambda_1 \quad \text{for } x \geq 0. \]
There exists $\lambda > 0$ such that, if $0 < \lambda_1 \leq \lambda$, $g$ attains its positive maximum and we get (3.1), with
\[ r = \left( \frac{q_c S^{q_c}}{2} \right)^{\frac{1}{q_c-1}} \quad \text{and} \quad R = |\Omega|^{-\frac{1}{2} + \frac{1}{q_c}} S \lambda, \]
thanks to (3.3). \hfill \blacksquare

Remark 3.1. Arguing as above we can see that there exists a constant $\alpha > 0$ such that
\[ F(u) \geq \alpha, \quad \text{for all } \|u\|_{H^2_0} = r. \]

Proposition 3.1. Let $R$ and $r$ be given by Lemma 3.1. Suppose that $f \neq 0$ and
\begin{equation}
\max \left( \|f\|_2, \|f\|_q \right) < \min(R', R),
\end{equation}
where
\[ R' = \frac{4 q_c S^{q_c}}{N \left( 2 (q_c - 1) \right)^{\frac{q}{q}}} . \]
Then there exists a function $u_1 \in H^2_0(\Omega)$ such that
\begin{equation}
F(u_1) = \min_{B_r} F(v) < 0,
\end{equation}
where $B_r$ is a ball of radius $r$. \hfill \blacksquare
where
\[ B_r = \left\{ v \in H^2_0, \|v\|_{H^2_0(\Omega)} < r \right\}, \]
and \( u_1 \) is a solution to \((P_{1,f})\). Moreover, \( u_1 \geq 0 \) whenever \( f \geq 0 \).

**Proof:** Without loss of generality, we can suppose that \( f(a) > 0 \) for some \( a \in \Omega \).

Let
\[ u_\varepsilon(x) = \frac{\varepsilon^{\frac{n-4}{2}} \phi(x)}{\left( \varepsilon + |x-a|^2 \right)^{\frac{n-4}{2}}}, \quad \varepsilon > 0, \]
where \( \phi \in C^\infty_0(\Omega) \) is a fixed function such that \( 0 \leq \phi \leq 1 \) and \( \phi \equiv 1 \) in some neighbourhood of \( a \).

Since
\[ \int_\Omega f u_\varepsilon \, dx > 0, \quad \text{for a small } \varepsilon, \]
we can choose \( t > 0 \) sufficiently small such that
\[ F(t u_\varepsilon) < 0. \]

Hence
\[ (3.6) \quad \inf_{B_r} F(v) < 0. \]

Let \( \{ u_m \} \) be a minimizing sequence of \((3.6)\). From \((3.4)\) and Lemma 3.1 we may assume that
\[ (3.7) \quad \|u_m\|_{H^2_0} < r_0 < r. \]

According to the Ekeland variational principle \([5]\) we may assume
\[ (3.8) \quad \Delta^2 u_m - |u_m|^q u - f \to 0 \quad \text{in } H^{-2}_0(\Omega). \]

On the other hand, from \((2.3)\) and \((3.4)\), we get
\[ (3.9) \quad \frac{1}{N} S^{\frac{N}{2}} - K > 0. \]

We deduce, from \((3.8)-(3.9)\) and Proposition 2.1, that \( \{ u_m \} \) has a subsequence converging to \( u_1 \in H^2_0(\Omega) \), and \( u_1 \) is a weak solution to \((P_{1,f})\).

Now we suppose that \( f \geq 0 \). Let \( v \in H^2_0(\Omega) \) be a solution to the following problem
\[ -\Delta v = |\Delta u_1|. \]
As in [10, 11] we get \( v > 0, \) \( v \geq |u_1| \) in \( \Omega, \)
\[
\int_\Omega |\nabla v|^2 = \int_\Omega |\nabla u_1|^2 \quad \text{and} \quad \int_\Omega |v|^q \geq \int_\Omega |u_1|^q .
\]
It then follows that
\[
F(v) \leq F(u_1) \quad \text{and} \quad \|v\|_{H_0^2} \leq r .
\]
Consequently \( F \) is minimized by a positive function.

This method allows us under suitable conditions on \( f \) and \( \lambda, \) to prove the existence of solutions to \((P_{\lambda,f}).\)

**Theorem 3.1.** Suppose that \( f \neq 0, \) then there exists \( \lambda_f > 0 \) such that if the following condition is satisfied

\[
0 < \lambda_f < \lambda^{-\frac{1}{w-2}} < \min \left( \frac{1}{\|f\|_2}, \frac{1}{\|f\|_q} \right) \min(R', R) , \tag{3.10}
\]
Problem \((P)_{\lambda,f}\) has at least one solution \( u_\lambda . \) Moreover \( u_\lambda \geq 0 \) whenever \( f \geq 0 . \)

**Proof:** For the proof we consider Problem \((P_{1,g})\) where \( g = f \lambda^{-\frac{1}{w-2}} . \)
Condition (3.10) implies that \( g \) satisfies (3.4). So the existence follows immediately from Proposition 3.1.

Now suppose, on the contrary, that \( u_\lambda \) exists for any \( \lambda \) such that
\[
0 < \lambda^{-\frac{1}{w-2}} < \min \left( \frac{1}{\|f\|_2}, \frac{1}{\|f\|_q} \right) \min(R', R) .
\]
Note that, since \( \lambda^{-\frac{1}{w-2}} u_\lambda \) is the solution to \((P_{1,g})\) obtained by (3.5), we have
\[
\|u_\lambda\|_{H_0^2(\Omega)} \leq r \lambda^{-\frac{1}{w-2}} .
\]
It follows from this that \( \|u_\lambda\|_{H_0^2(\Omega)} \to 0 \) as \( \lambda \downarrow 0 . \)
Passing to the limit in \((P_{\lambda,f})\) we deduce that \( f \equiv 0, \) which yields to a contradiction. ■

4 – Existence of a second solution

In this section we shall show, under additional conditions that \((P_{\lambda,f})\) possesses a second solution. Here we use the mountain pass theorem without the Palais–Smale condition [2, 8]. As in the preceding section, we first deal with the case \( \lambda = 1 . \)
Assume that condition (3.4) is satisfied and that \( f > 0 \) in some neighbourhood of \( \alpha \). Set
\[
v_\varepsilon = \frac{u_\varepsilon}{\|u_\varepsilon\|_{q_c}}.
\]
The main result of this section is the following.

**Theorem 4.1.** There exists \( t_0 > 0 \) such that if \( f \) satisfies
\[
\|f\|_q^q < \frac{t_0}{K_1} \int_\Omega f v_\varepsilon \, dx,
\]
where
\[
K_1 = \frac{N \frac{q}{q} \|X\|_{q_c}^q}{2 q (4 q_c)^{\frac{q}{q}}},
\]
then \((P_{1,f})\) has at least two distinct solutions.

**Proof:** The proof relies on a variant of the mountain pass theorem without the (PS) condition. We have, for \( \varepsilon \) sufficiently small (see [4]),
\[
\|\Delta v_\varepsilon\|_2^2 = S + O(\varepsilon^{\frac{N-4}{4}}).
\]
Set
\[
h(t) = F(t v_\varepsilon) = \frac{1}{2} t^2 X_\varepsilon - \frac{1}{q_c} t^{q_c} - t \int_\Omega f v_\varepsilon \, dx \quad \text{for} \quad t \geq 0,
\]
where \( X_\varepsilon = \|\Delta v_\varepsilon\|_2^2 \).

Since \( h(t) \) goes to \( -\infty \) as \( t \) goes to \( +\infty \), \( \sup_{t \geq 0} h(t) \) is achieved at some \( t_\varepsilon \geq 0 \). Remark 3.1 asserts that \( t_\varepsilon > 0 \), and we deduce
\[
h'(t_\varepsilon) = t_\varepsilon (X_\varepsilon - t_\varepsilon^{q_c-2}) - \int_\Omega f v_\varepsilon \, dx = 0 \quad \text{and} \quad h''(t_\varepsilon) \leq 0,
\]
thus
\[
\left( \frac{1}{q_c - 1} \right)^{\frac{1}{q_c-2}} X_\varepsilon^{\frac{1}{q_c-2}} \leq t_\varepsilon \leq X_\varepsilon^{\frac{1}{q_c-2}}.
\]
Let \( t_0 = \frac{1}{2} \left( \frac{1}{q_c-1} \right)^{\frac{1}{q_c-2}} S^{\frac{1}{q_c-2}} \). We deduce from (4.2) and (4.4) that, for \( \varepsilon_0 \) small,
\[
t_0 < t_\varepsilon \quad \text{for} \quad \varepsilon \in (0, \varepsilon_0).
\]
Thus
\[
\sup_{t \geq t_0} h(t) = \sup_{t \geq t_0} h(t).
\]
On the other hand, since the function \( t \rightarrow \frac{1}{2} t^2 X_e - \frac{1}{q_e} t^q_e \) is increasing on the interval \([0, X_e^{-q_e-2}]\), we get
\[
h(t_e) \leq \frac{2}{N} S^N - t_e \int_{\Omega} f v_e \, dx + O(\varepsilon^{N-2}) ,
\]
thanks to (4.2). Hence
\[
(4.6) \quad h(t_e) \leq \frac{2}{N} S^N - t_0 \int_{\Omega} f v_e \, dx + O(\varepsilon^{N-2}) .
\]
Consequently if we let
\[
(4.7) \quad t_0 \int_{\Omega} f v_e \, dx > K_1 \| f \|_q^q ,
\]
we deduce that
\[
(4.8) \quad \sup_{t \geq 0} F(t v_e) < \frac{2}{N} S^N - K .
\]
Note that there exists \( t_1 \) large enough such that
\[
(4.9) \quad F(t_1 v_e) < 0 \quad \text{and} \quad \| t_1 v_e \|_{H^2} > r ,
\]
where \( r \) is given by Lemma 3.1. Hence
\[
\alpha \leq c_2 = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} F(\gamma(s)) < \frac{2}{N} S^N - K ,
\]
where
\[
\Gamma = \left\{ \gamma \in C\left([0,1], H^2_\theta(\Omega)\right) : \gamma(0) = 0, \ \gamma(1) = t_1 v_e \right\} ,
\]
provided \( \varepsilon \) is small enough. Then, according to the mountain pass theorem without the (PS) condition, there exists a sequence \( \{u_m\} \) in \( H^2_\theta(\Omega) \) such that
\[
F(u_m) \to c_2 \quad \text{and} \quad dF(u_m) \to 0 \quad \text{in} \quad H^{-2}_\theta(\Omega) .
\]
Since \( c_2 < \frac{2}{N} S^N - K \), we deduce from Proposition 2.1 that there exists \( u_2 \) such that \( c_2 = F(u_2) \) and \( u_2 \) is a weak solution to \((P_{1,f})\).

This solution is distinct from \( u_1 \) since \( c_1 < 0 < c_2 \). So the proof is complete.

Finally, by using Theorem 4.1, we deduce the

**Corollary 4.1.** Assume (3.10). If
\[
\lambda_{q_e-2}^{q_e-1} < \frac{t_0}{K_1 \| f \|_q^q} \int_{\Omega} f v_e \, dx ,
\]
for \( \varepsilon \) small enough, then problem \((P_{\lambda,f})\) has at least two solutions.
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