WEIGHTED NORM INEQUALITY
FOR THE POISSON INTEGRAL ON THE SPHERE

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Abstract: We obtain, for each $p$, $1 < p < \infty$, a necessary and sufficient condition for the Poisson integral of functions defined on the sphere $S^n$, to be bounded from a weighted space $L^p(S^n, Wd\sigma)$ into a space $L^p(B, \nu)$, where $\sigma$ is the Lebesgue measure on $S^n$ and $\nu$ is a positive measure on the unit ball $B$ of $\mathbb{R}^{n+1}$.

Introduction

In this paper we consider a homogeneous space $X = G/H$ where $G$ is a locally compact Hausdorff topological group and $H$ is a compact subgroup of $G$ which is provided with a quasi-distance $d$ and with a measure $\mu$ induced on $X$ by a Haar measure on the topological group $G$. If $x \in X$ and $r > 0$, $B(x, r)$ will denote the ball $\{ y \in X : d(x, y) < r \}$ in $X$. We also write $\check{X} = X \times [0, \infty)$ and if $B = B(x, r)$ we write $\check{B} = B(x, r) \times [0, r]$.

We define the maximal operator $\mathcal{M}$ by

$$\mathcal{M}f(x, r) = \sup_{s \geq r} \frac{1}{\mu(B(x, s))} \int_{B(x, s)} |f(y)| \, d\mu(y)$$

for all real-valued locally integrable function $f$ on $X$ and $(x, r) \in \check{X}$. If $r = 0$ the above supremum is taken over all $s > 0$ and $\mathcal{M}f(x, 0) = f^*(x)$ is the Hardy-Littlewood maximal function.

A weight is a positive locally integrable function $W(x)$ on $X$ and we will write $W(A) = \int_A W \, d\mu$. We say that $W$ is a weight in the class $A_\infty(X)$ if there exist

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positive constants $C_W$ and $\delta$ such that
\[
\frac{\mu(A)}{\mu(B)} \leq C_W \left( \frac{W(A)}{W(B)} \right)^\delta,
\]
for all ball $B = B(x, r), \ x \in X, \ r > 0$, and all Borel subsets $A$ of $B$. We observe that the above inequality is equivalent to a similar one where $\mu$ appears instead of $W$ and conversely (see [5, 1]). We write $L^p(W) = L^p(X, W(x) \, d\mu(x))$, $1 \leq p < \infty$.

Let $1 < p < \infty$, $p'$ such that $1/p + 1/p' = 1$, let $\beta$ be a positive measure on the Borel subsets of $\tilde{X}$ and $W$ a weight on $X$. In Section 2 we introduce a maximal operator of dyadic type $M_b$, where $b$ is an integer, using partitions of dyadic type for the homogeneous space $X$ introduced in Section 1.

In Section 3 we prove the following theorem.

**Theorem 3.1.** Let $G$ be a compact or an Abelian group, let $1 < p < \infty$ and let $W$ be a weight on $X$ such that $W^{1-p'} \in A_\infty(X)$. Then the following conditions are equivalent:

(i) There exists a constant $C > 0$, such that, for all $f \in L^p(W)$,
\[
\int_X [Mf(x, r)]^p \, d\beta(x, r) \leq C \int_X |f(x)|^p W(x) \, d\mu(x).
\]

(ii) There exists a constant $C > 0$, such that, for all balls $B = B(z, t)$, $0 \leq t < \infty$,
\[
\int_B [M(W^{1-p'} \chi_B)(x, r)]^p \, d\beta(x, r) \leq C \int_B W^{1-p'}(x) \, d\mu(x) < \infty.
\]

The above result for $X = \mathbb{R}^n$ was proved in Ruiz-Torrea [7]. A similar result for the fractional maximal operator was obtained in Bernardis-Salinas [1]. The condition (ii) of Theorem 3.1 implies the condition
\[
\frac{\beta(B)^{1/p}}{\mu(B)} \left( \int_B W^{1-p'}(x) \, d\mu(x) \right)^{1/p'} \leq C < \infty
\]
for all balls $B$. It was proved in Ruiz-Torrea [8] that the above condition is a necessary and sufficient condition for $M$ to be a bounded operator from $L^p(X, W(x) \, d\mu(x))$ into weak $-L^p(\tilde{X}, \beta)$. In the particular case $W(x) \equiv 1$, the condition (ii) of Theorem 3.1 is equivalent to the Carleson’s condition for the homogeneous space $X$:
\[
\beta(B) \leq C \mu(B)
\]
for all balls $B$ and for a constant $C > 0$. 
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Now, if \( x \in \mathbb{R}^{n+1} \), we write \(|x| = (x \cdot x)^{1/2}\) and \( d(x, y) = |x - y|\), where \( x \cdot y \) is the usual scalar product of \( x \) and \( y \) in \( \mathbb{R}^{n+1} \). Here \( S^n \) will denote the unit \( n \)-sphere \( \{ y \in \mathbb{R}^{n+1} : |y| = 1 \} \) in \( \mathbb{R}^{n+1} \), \( \sigma \) the normalized Lebesgue measure on \( S^n \) and \( h : [1 - \sqrt{2}, 1] \to [0, 2] \) will be the function defined by \( h(r) = \sqrt{2}(1 - r) \).

The Poisson kernel for the sphere \( S^n \) is given by

\[
P_{ry}(x) = \frac{1}{\omega_n} \frac{1 - r^2}{|ry - x|^{n+1}}
\]

for \( x, y \in S^n \) and \( 0 \leq r < 1 \), where \( \omega_n \) is the area of the sphere \( S^n \). For a real-valued integrable function \( f \) we denote by \( u_f(ry) \) the Poisson integral

\[
u_f(ry) = \int_{S^n} P_{ry}(x) f(x) \, d\sigma(x).
\]

and we define the maximal function \( u_f^* \) by

\[
u_f^*(ry) = \sup_{0 \leq s \leq r} |u_f(sy)|, \quad 0 \leq r < 1, \quad y \in S^n.
\]

If \( B \) is the open ball \( B(z, t) = \{ x \in S^n : |x - z| < t \} \), \( 0 < t \leq 2 \), we define

\[
\hat{B} = \{ sx : h^{-1}(t) \leq s \leq 1, \ x \in B \} \quad \text{if} \quad 0 < t \leq \sqrt{2};
\]

\[
\check{B} = \{ sx : 0 \leq s \leq 1, \ x \in B \} \quad \text{if} \quad \sqrt{2} \leq t \leq 2.
\]

We observe that \( \hat{B} \) is a truncated cone in the ball \( B = \{ y \in \mathbb{R}^{n+1} : |y| \leq 1 \} \) in \( \mathbb{R}^{n+1} \) if \( 0 < t \leq \sqrt{2} \) and a cone if \( \sqrt{2} \leq t \leq 2 \).

In Section 4 we prove the following result.

**Theorem 4.1.** Let \( 1 < p < \infty \), let \( W \) be a weight on \( S^n \) such that \( W^{1-n'/p} \in A_\infty(S^n) \) and let \( \nu \) be a Borel positive measure on \( S^n \). Then the following conditions are equivalent:

(i) There exists a constant \( C > 0 \), such that, for all \( f \in L^p(W) \),

\[
\int_{\hat{B}} [u_f^*(y)]^p \, d\nu(y) \leq C \int_{S^n} |f(x)|^p W(x) \, d\sigma(x).
\]

(ii) There exists a constant \( C > 0 \), such that, for all balls \( B = B(z, t) \), \( 0 < t \leq 2 \),

\[
\int_{\hat{B}} [u_{W^{1-n'}}_{\hat{B}}(y)]^p \, d\nu(y) \leq C \int_B W^{1-n'}(x) \, d\sigma(x) < \infty.
\]

We point out that the Theorem 4.1 for \( W \equiv 1 \) and \( n = 1 \) was proved in Carleson [2].
1 – Preliminaries

In this section we introduce some notations, definitions and basic facts.

Let $G$ be a locally compact Hausdorff topological group with unit element $e$, $H$ be a compact subgroup and $\pi : G \to G/H$ the canonical map. Let $dg$ denote a left Haar measure on $G$, which we assume to be normalized in the case of $G$ to be compact. If $A$ is a Borel subset of $G$ we will denote by $|A|$ the Haar measure of $A$. The homogeneous space $X = G/H$ is the set of all left cosets $\pi(g) = gH, g \in G$, provided with the quotient topology. The Haar measure $dg$ induces a measure $\mu$ on the Borel $\sigma$-field on $X$. For $f \in L^1(X)$,

$$\int_X f(x) \, d\mu(x) = \int_G f \circ \pi(g) \, dg.$$  

We observe that the group $G$ acts transitively on $X$ by the map $(g, \pi(h)) \mapsto g\pi(h) = \pi(gh)$, that is, for all $x, y \in X$, there exists $g \in G$ such that $gx = y$. We also observe that the measure $\mu$ on $X$ is invariant on the action of $G$, that is, if $f \in L^1(X), g \in G$ and $R_g f(x) = f(g^{-1}x)$, then

$$\int_X f(x) \, d\mu(x) = \int_X R_g f(x) \, d\mu(x).$$

**Definition 1.1.** A quasi-distance on $X$ is a map $d : X \times X \to [0, \infty)$ satisfying:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(gx, gy) = d(x, y)$ for all $g \in G, x, y \in X$;

(iv) there exists a constant $K \geq 1$ such that, for all $x, y, z \in X$,  

$$d(x, y) \leq K[d(x, z) + d(z, y)];$$

(v) the balls $B(x, r) = \{y \in X : d(x, y) < r\}, x \in X, r > 0$, are relatively compact and measurable, and the balls $B(\mathbb{1}, r), r > 0$, form a basis of neighborhoods of $\mathbb{1} = \pi(e)$;

(vi) there exists a constant $A \geq 1$ such that, for all $r > 0$ and $x \in X$,

$$\mu(B(x, 2r)) \leq A\mu(B(x, r)).$$

\[ \Box \]
In this paper $X$ will denote a homogeneous space provided with a quasi-distance $d$.

Given a quasi-distance $d$ on $X$, there exists a distance $\rho$ on $X$ and a positive real number $\gamma$ such that $d$ is equivalent to $\rho^\gamma$ (see [5]). Therefore the family of $d$-balls is equivalent to the family of $\rho^\gamma$-balls and $\rho^\gamma$-balls are open sets.

It follows by Definition 1.1(iii) that $B(gx, r) = gB(x, r)$ for all $g \in G$, $x \in X$ and $r > 0$, and hence $\mu(B(gx, r)) = \mu(B(x, r))$. Thus we can write $X = \bigcup_{j \geq 1} g_j B(x, r)$ where $(g_j)$ is a sequence of elements of $G$ and consequently $\mu(B(x, r)) > 0$. In particular, $X$ is separable.

**Lemma 1.1.** Let $b$ be a positive integer and let $\lambda = 8K^5$. Then for each integer $k$, $-b \leq k \leq b$, there exist an enumerable Borel partition $\mathcal{A}_b^k$ of $X$ and a positive constant $C$ depending only on $X$, such that:

(i) for all $Q \in \mathcal{A}_b^k$, $-b \leq k \leq b$, there exists $x_Q \in Q$ such that

$$B(x_Q, \lambda^k) \subset Q \subset B(x_Q, \lambda^{k+1})$$

and

$$\mu(B(x_Q, \lambda^{k+1})) \leq C \mu(Q) ;$$

(ii) if $-b \leq k < b$, $Q_1 \in \mathcal{A}_b^{k+1}$, $Q_2 \in \mathcal{A}_b^k$ and $Q_1 \cap Q_2 \neq \emptyset$, then $Q_2 \subset Q_1$, and

$$0 < \mu(Q_1) \leq C \mu(Q_2) ;$$

(iii) for all $x \in X$ and $r$, $\lambda^{-b-1} \leq r \leq \lambda^b$, there exist $Q \in \mathcal{A}_b^k$ for some $-b \leq k \leq b$ and $g \in G$ such that $d(gx, x) \leq \lambda^{k+1}$, $B(x, r) \subset gQ$ and

$$\mu(Q) \leq C \mu(B(x, r)) .$$

**Proof:** The properties (i) and (ii) follow by Lemma 3.21, p. 852 of [9] and by Definition 1.1.

Let us prove (iii). Given $x \in X$ and $\lambda^{-b-1} \leq r \leq \lambda^b$, let $-b \leq k \leq b$ such that $\lambda^{k-1} \leq r \leq \lambda^k$. There exists an unique $Q \in \mathcal{A}_b^k$ such that $x \in Q$. Consider $x_Q$ as in (i) and $g \in G$ such that $x = g x_Q$. If $a$ is an integer such that $2^{a-1} < \lambda \leq 2^a$, then by (i) we have

$$B(x, r) \subset B(g x_Q, \lambda^k) \subset gQ \subset B(x, \lambda^{k+1})$$
and hence by Definition 1.1(vi) we have

\[ \mu(Q) \leq \mu(B(x_Q, 2^a \lambda^k)) \]
\[ \leq A^a \mu(B(x_Q, \lambda^k)) \]
\[ \leq A^{2a} \mu(B(x, r)). \]

We also have that \( d(gx, x) = d(x, x_Q) \leq \lambda^{k+1}. \]

Let \((\Omega, \mathcal{F}, \nu)\) be a \(\sigma\)-finite measure space, let \((\mathcal{F}_k)_{k \in \mathbb{Z}}\) be an increasing sequence of sub-\(\sigma\)-fields of \(\mathcal{F}\), and for each \(k \in \mathbb{Z}\), consider a real-valued \(\mathcal{F}_k\)-measurable function \(f_k\). We say that the sequence \((f_k)_{k \in \mathbb{Z}}\) is a martingale with respect to the sequence \((\mathcal{F}_k)_{k \in \mathbb{Z}}\) if, for all \(k \in \mathbb{Z}\) and all \(A \in \mathcal{F}_k\) such that \(\nu(A) < \infty\), we have that

\[ \int_A |f_k| \, d\nu < \infty, \quad \int_A f_k \, d\nu = \int_{A} f_{k+1} \, d\nu. \]

Now, consider a \(\sigma\)-finite measure \(\nu\) on the Borel \(\sigma\)-field of \(X\) and let \(\mathcal{F}_k\) be the \(\sigma\)-field generated by the partition \(A^b_{-k}\) for \(-b \leq k \leq b\), by \(A^b_{-b}\) for \(k \geq b\) and by \(A^b_{b}\) for \(k \leq -b\). If \(f \in L^1(X, \nu)\),

\[ f_k(x) = E[f|\mathcal{F}_k](x) = \sum_{Q \in A^b_{k}} \left( \frac{1}{\nu(Q)} \int_Q f(y) \, d\nu(y) \right) \chi_Q(x), \quad -b \leq k \leq b; \]

and \(f_k = f_b\) for \(k \geq b\), \(f_k = f_{-b}\) for \(k \leq -b\), then \((f_k)_{k \in \mathbb{Z}}\) is a martingale with respect to the sequence \((\mathcal{F}_k)_{k \in \mathbb{Z}}\). We define the maximal operator \(M^b\), for all \(f \in L^1(X, \nu)\) by

\[ M^b f(x) = \sup_{k \in \mathbb{Z}} E[|f| \mid \mathcal{F}_k](x) = \sup_{Q \in A^b_{k}} \frac{1}{\nu(Q)} \int_Q |f(y)| \, d\nu(y). \]

where \(A^b = \bigcup_{-b \leq k \leq b} A^b_{k}\).

The next result can be found in Dellacherie-Meyer [3], number 40, p. 37.

**Theorem 1.1.** If \(1 < p < \infty\) and \(f \in L^p(X, \nu)\), then

\[ \|M^b f\|_{L^p(X, \nu)} \leq p^{1/p} \|f\|_{L^p(X, \nu)}. \]
2 – A maximal operator of dyadic type

Let $b$ be a fixed positive integer. Given $Q \in \mathcal{A}^b = \bigcup_{-b \leq k \leq b} A_k^b$, where $A_k^b$ are the partitions of $X$ in Lemma 1.1, $Q$ will denote the subset $Q \times [0, \alpha^{-1}(\mu(Q))]$ of $\tilde{X} = X \times [0, \infty)$, where $\alpha : [0, \infty) \to [0, \infty)$ is the function defined by $\alpha(r) = \mu(B(\mathbb{1}, r))$, $\mathbb{1} = \pi(e)$.

If $f$ is a real-valued locally integrable function on $X$, we define, for each $(x, r) \in \tilde{X}$,

$$M_d^b f(x, r) = \sup_{x \in Q \in A^b} \frac{1}{\mu(Q)} \int_Q |f(y)| \, d\mu(y).$$

If $\mu(Q) < \alpha(r)$ for all $Q \in A^b$ such that $x \in Q$, we define $M_d^b f(x, r) = 0$.

**Lemma 2.1.** Let $W$ be a weight and let $A$ be a measurable subset of $X$. If $1 < p < \infty$ and $W^{-1} \chi_A \not\in L^p(W)$, then there exists a positive function $f \in L^p(W)$ such that $\int_A f(x) \, d\mu(x) = \infty$.

**Proof:** Let $\psi$ be the linear functional on $L^p(W)$ given by $\psi(g) = \int_A g \, d\mu$. Since $W^{-1} \chi_A \not\in L^p(W)$, it follows by the Riesz representation theorem that $\psi$ is not continuous. Therefore, there exists $\varepsilon > 0$, such that, for each positive integer $m$, there exists $g_m \in L^p(W)$ such that $\|g_m\|_{L^p(W)} \leq 2^{-m}$ and $|\psi(g_m)| \geq \varepsilon$.

We set $f_m(x) = |g_1(x)| + \cdots + |g_m(x)|$ and then, for all $m, k \geq 1$,

$$\|f_{m+k} - f_m\|_{L^p(W)} \leq \|g_{m+1}\|_{L^p(W)} + \cdots + \|g_{m+k}\|_{L^p(W)} < 2^{-m}.$$

Hence $(f_m)$ is a Cauchy sequence in $L^p(W)$ and therefore there exists $f \in L^p(W)$ such that $f_m \to f$ in $L^p(W)$. On the other hand

$$\psi(f_m) \geq |\psi(g_1)| + \cdots + |\psi(g_m)| \geq m \varepsilon.$$

But $f_m \uparrow f$ a.e. and thus by the monotone convergence theorem we obtain

$$\int_A f \, d\mu = \lim_{m \to \infty} \psi(f_m) = \infty.$$

**Theorem 2.1.** Given a weight $W$ on $X$, a positive measure $\beta$ on $\tilde{X}$, and $1 < p < \infty$, the following conditions are equivalent:
(i) There exists a constant $C > 0$, such that, for all $f \in L^p(W)$ and all positive integer $b$,
\[
\int_X |M_d^b f(x,r)|^p \, \, d\beta(x,r) \leq C \int_X |f(x)|^p \, W(x) \, \, d\mu(x) .
\]

(ii) There exists a constant $C > 0$, such that, for all $Q \in A^b$ and all positive integer $b$,
\[
\int_Q |M_d^b (W^{1-p'}(x,r))|^p \, \, d\beta(x,r) \leq C \int_Q W^{1-p'}(x) \, \, d\mu(x) < \infty .
\]

**Proof:** The proof of (i)⇒(ii) is exactly as the proof of (i)⇒(ii) in Theorem 3.1.

Proof of (ii)⇒(i): Let us fix $f \in L^p(W)$ and for each $k \in \mathbb{Z}$, let $\Omega_k$ be the set
\[
\Omega_k = \left\{ (x,r) \in \tilde{X} : M_d^b f(x,r) > 2^k \right\} .
\]

For each $k \in \mathbb{Z}$, we denote by $C_k^0$ the family formed by all $Q \in A^b$ such that
\[
|f|_Q = \frac{1}{\mu(Q)} \int_Q |f(y)| \, \, d\mu(y) > 2^k .
\]

Since for every $Q \in A_k^b$, $-b \leq k < b$, there exists $Q' \in A_{k+1}^b$ such that $Q \subset Q'$, then every element $Q \in C_k^0$ is contained in a maximal element $Q' \in C_k^0$. We denote by $C_k$ the family $\{Q^k_j : j \in J_k\}$ formed by all maximal elements $Q \in C_k^0$. Since $A_k^b$ is a partition of $X$ and all elements of $C_k$ are maximal, we can conclude that the sets $Q^k_j$, $j \in J_k$, are pairwise disjoint. Therefore the sets $\tilde{Q}^k_j$, $j \in J_k$, are also pairwise disjoint and,
\[
\Omega_k = \bigcup_{j \in J_k} \tilde{Q}^k_j .
\]

Now, for each $k \in \mathbb{Z}$ and each $j \in J_k$, let
\[
E^k_j = \tilde{Q}^k_j \setminus \Omega_{k+1} .
\]

Then the sets $E^k_j$ and $E^h_i$ are disjoint for $(k,j) \neq (h,i)$ and
\[
\left\{ (x,r) : M_d^b f(x,r) > 0 \right\} = \bigcup_{k \in \mathbb{Z}} (\Omega_k \setminus \Omega_{k+1}) = \bigcup_{k \in \mathbb{Z}} \bigcup_{j \in J_k} E^k_j .
\]
Therefore
\[ \int_X [M_d^bf(x, r)]^p \, d\beta(x, r) = \sum_{k,j} \int_{E^k_j} [M_d^bf(x, r)]^p \, d\beta(x, r) \]
\[ \leq 2^p \sum_{k,j} \beta(E^k_j) (2^k)^p \]
(2.1)
\[ \leq 2^p \sum_{k,j} \beta(E^k_j) \left( \frac{1}{\mu(Q^k_j)} \int_{Q^k_j} |f(x)| \, d\mu(x) \right)^p. \]

Now, we introduce the following notations:
\[ \nu(x) = W^{1-p'}(x), \quad \nu(A) = \int_A \nu(x) \, d\mu(x), \]
\[ \gamma_{k,j} = \beta(E^k_j) \left( \frac{\nu(Q^k_j)}{\mu(Q^k_j)} \right)^p, \quad g_{k,j} = \left( \frac{1}{\nu(Q^k_j)} \int_{Q^k_j} |f(x)| \, \nu(x) \, d\mu(x) \right)^p, \]
\[ Y = \{(k,j) : k \in \mathbb{Z}, j \in J_k \}, \quad \Gamma(\lambda) = \{(k,j) \in Y : g_{k,j} > \lambda \}. \]

Let \( \gamma \) be the measure on \( Y \) such that \( \gamma(\{(k,j)\}) = \gamma_{k,j} \) and let \( g \) be the function defined on \( Y \) by \( g((k,j)) = g_{k,j} \). We have that
\[ \gamma_{k,j} g_{k,j} = \beta(E^k_j) \left( \int_{Q^k_j} |f(x)| \, d\mu(x) \right)^p \]
and hence it follows by (2.1) that
\[ \int_X [M_d^bf(x, r)]^p \, d\beta(x, r) \leq 2^p \sum_{k,j} \gamma_{k,j} g_{k,j} \]
\[ = 2^p \int_0^\infty \gamma(\Gamma(\lambda)) \, d\lambda \]
\[ = 2^p \int_0^\infty \left( \sum_{(k,j) \in \Gamma(\lambda)} \gamma_{k,j} \right) \, d\lambda \]
(2.2)
\[ = 2^p \int_0^\infty \sum_{(k,j) \in \Gamma(\lambda)} \int_{E^k_j} \left( \frac{\nu(Q^k_j)}{\mu(Q^k_j)} \right)^p \, d\beta(x, r) \, d\lambda. \]

For each \( \lambda > 0 \), let \( \{Q^\lambda_i : i \in I_\lambda \} \) be the family formed by all maximal elements of the family
\[ \{Q^k_j : (k,j) \in \Gamma(\lambda) \} = \left\{ Q^k_j : \frac{1}{\nu(Q^k_j)} \int_{Q^k_j} |f(x)| \, \nu(x) \, d\mu(x) > \lambda^{1/p} \right\}. \]
If \( Q_j^k \subset Q_i^\lambda \) and \( (x, r) \in E_j^k \), then \( x \in Q_j^k, \mu(Q_j^k) \geq \alpha(r) \) and thus

\[
\mathcal{M}_d^b(\nu \chi_{Q_i^\lambda})(x, r) = \sup_{\substack{Q \in \mathcal{A}^b \\mu(Q) \geq \alpha(r) \\nu(Q) \geq \mu(Q)}} \frac{\nu(Q \cap Q_i^\lambda)}{\mu(Q)} \geq \frac{\nu(Q_j^k)}{\mu(Q_j^k)}.
\]

Therefore, if \( Q_j^k \subset Q_i^\lambda \) we obtain

\[
\int_{E_j^k} \left( \frac{\nu(Q_j^k)}{\mu(Q_j^k)} \right)^p d\beta(x, r) \leq \int_{E_j^k} \left[ \mathcal{M}_d^b(\nu \chi_{Q_i^\lambda})(x, r) \right]^p d\beta(x, r).
\]

Taking into account that the sets \( E_j^k \) are disjoint, it follows from (2.2), (2.3) and by the hypothesis that

\[
\int_X |M_d^b f(x, r)|^p d\beta(x, r) \leq 2^p \int_0^\infty \sum_{i \in \mathbb{I}_X} \sum_{Q_j^k \subset Q_i^\lambda} \int_{E_j^k} \left[ \mathcal{M}_d^b(\nu \chi_{Q_i^\lambda})(x, r) \right]^p d\beta(x, r)
\]

\[
\leq 2^p \int_0^\infty \sum_{i \in \mathbb{I}_X} \sum_{Q_j^k \subset Q_i^\lambda} \left[ \mathcal{M}_d^b(\nu \chi_{Q_i^\lambda})(x, r) \right]^p d\beta(x, r)
\]

\[
\leq C 2^p \int_0^\infty \sum_{i \in \mathbb{I}_X} \sum_{Q_j^k \subset Q_i^\lambda} \nu(x) d\mu(x)
\]

\[
= C 2^p \int_0^\infty \nu \left( \bigcup_{(k, j) \in \Gamma(\lambda)} Q_j^k \right) d\mu(x).
\]

It follows by the definition of the maximal operator \( M_d^b \) in Section 1 and by the definition of \( \Gamma(\lambda) \) that

\[
\bigcup_{(k, j) \in \Gamma(\lambda)} Q_j^k \subset \left\{ x \in X : M_d^b \left( \frac{|f|}{\nu} \right)(x) > \lambda^{1/p} \right\}.
\]

Then, by (2.4), (2.5) and Theorem 1.1,

\[
\int_X |M_d^b f(x, r)|^p d\beta(x, r) \leq C 2^p \int_0^\infty \nu \left( \left\{ x : \left( M_d^b \left( \frac{|f|}{\nu} \right)(x) \right)^p > \lambda \right\} \right) d\lambda
\]

\[
= C 2^p \int_X \left( M_d^b \left( \frac{|f|}{\nu} \right)(x) \right)^p \nu(x) d\mu(x)
\]

\[
\leq C 2^p (p')^p \int_X \frac{|f(x)|^p}{\nu(x)} \nu(x) d\mu(x)
\]

\[
= C 2^p (p')^p \int_X |f(x)|^p \nu(x) d\mu(x).
\]
Remark 2.1. Let us fix $g \in G$ and let $g^{-1}A^b_k = \{g^{-1}Q : Q \in A^b_k\}$, $g^{-1}A^b = \{g^{-1}Q : Q \in A^b\}$. Then for each $-b \leq k \leq b$, $g^{-1}A^b_k$ is a partition of $X$ and Lemma 1.1 and Theorem 1.1 in Section 1 also hold, with the same constants, when we change $A^b_k$ for $g^{-1}A^b_k$. If $f$ is a real-valued locally integrable function on $X$, we define

$$M_{d}^{b,g}f(x,r) = \sup_{x \in Q \in g^{-1}A^b} \frac{1}{\mu(Q)} \int_{Q} |f(y)| \, d\mu(y) .$$

Then

$$M_{d}^{b}(R_g f)(gx, r) = M_{d}^{b,g}f(x, r)$$

where $R_g f(x) = f(g^{-1}x)$. The Theorem 2.1 also hold, with the same proof, when we change the operator $M_{d}^{b}$ for $M_{d}^{b,g}$ and the family $A^b$ for $g^{-1}A^b$. □

3 – The boundedness of the operator $M$

Given a positive integer $b$ and a real-valued locally integrable function $f$ on $X$, we define for $(x, r) \in \tilde{X}$,

$$M^b f(x, r) = \sup_{\max\{\lambda^{-b-1} r \leq s \leq \lambda^b B(x,s)\}} \int_{B(x,s)} |f(y)| \, d\mu(y) .$$

We define $M^b f(x, r) = 0$ if $r > \lambda^b$ and we observe that $M^b f(x, r) \uparrow Mf(x, r)$ if $b \uparrow \infty$ for all $(x, r) \in \tilde{X}$.

Let us denote

$$G_b = \{g \in G : d(gx, x) \leq \lambda^{b+1} \text{ for all } x \in X\} .$$

If $d(g \mathbb{1}, \mathbb{1}) = d(gx, x)$ for all $x \in X$ and $g \in G$, in particular if $G$ is an Abelian group, then

$$G_b = \{g \in G : d(g \mathbb{1}, \mathbb{1}) \leq \lambda^{b+1}\} ,$$

and hence $G_b$ is relatively compact in $G$ and $0 < |G_b| < \infty$ (see [4]).

Lemma 3.1. Let $b$ be a positive integer, $g \in G$, let $M_{d}^{b,g}$ be the maximal operator defined in Remark 2.1, let $f$ be a real-valued locally integrable function on $X$ and let $(x, r) \in \tilde{X}$. Then

$$M_{d}^{b,g}f(x, r) \leq CMf(x, r) .$$
If $G$ is a compact or an Abelian group, then

\[(3.2) \quad \mathcal{M}^b f(x, r) \leq \frac{C}{|G_b|} \int_{G_b} \mathcal{M}_d^{b,g} f(x, r) \, dg .\]

The constants $C$ in (3.1) and in (3.2) depend only on $X$ and if $X$ is compact we can change $G_b$ for $G$.

**Proof:** Let us fix $(x, r) \in \bar{X}$ and $g \in G$. If $\mu(Q) < \alpha(r)$ for all $Q \in \mathcal{A}^b$ such that $x \in g^{-1}Q$, we have $\mathcal{M}_d^{b,g} f(x, r) = 0$. Thus to prove (3.1), it is enough to consider $Q \in \mathcal{A}_b^b$, $-b \leq k \leq b$, such that $x \in g^{-1}Q$ and $\mu(Q) \geq \alpha(r)$. By Lemma 1.1(i) there exist $x_Q \in Q$ such that $Q \subset B(x_Q, \lambda^{k+1})$ and $\mu(B(x_Q, \lambda^{k+1})) \leq C \mu(Q)$. For $t = 2K \lambda^{k+1}$ we have $B(g^{-1}x_Q, \lambda^{k+1}) \subset B(x, t)$ and hence

$$\alpha(t) = \mu(B(x, t)) \geq \mu(B(g^{-1}x_Q, \lambda^{k+1})) \geq \mu(Q) \geq \alpha(r).$$

If $2^{a-1} < K \leq 2^a$, it follows by Definition 1.1(vi) that

$$\mu(B(x, t)) \leq A^{a+1} \mu(B(x_Q, \lambda^{k+1})) \leq A^{a+1} C \mu(g^{-1}Q).$$

Therefore

$$\frac{1}{\mu(g^{-1}Q)} \int_{g^{-1}Q} |f(y)| \, d\mu(y) \leq \frac{A^{a+1} C}{\mu(B(x, t))} \int_{B(x, t)} |f(y)| \, d\mu(y) \leq A^{a+1} C \mathcal{M} f(x, r)$$

and hence we obtain (3.1).

Let us fix $(x, r) \in \bar{X}$. If $r > \lambda^b$ we have $\mathcal{M}^b f(x, r) = 0$ and thus we can suppose $r \leq \lambda^b$. Given $s$ such that, $\lambda^{-b-1} \leq s \leq \lambda^b$ and $s \geq r$, by Lemma 1.1(iii), there exist $Q \in \mathcal{A}_b^b$ for some $-b \leq k \leq b$ and $g \in G_b$, such that $B(x, s) \subset g^{-1}Q$ and $\mu(Q) \leq C \mu(B(x, s))$. Then

$$\frac{1}{\mu(B(x, s))} \int_{B(x, s)} |f(y)| \, d\mu(y) \leq \frac{C}{\mu(g^{-1}Q)} \int_{g^{-1}Q} |f(y)| \, d\mu(y) \leq C \mathcal{M}_d^{b,g} f(x, r)$$

since $\mu(Q) \geq \mu(B(x, s)) \geq \alpha(r)$. Therefore, integrating both sides of the above inequality on $G_b$, we have that

$$\frac{1}{\mu(B(x, s))} \int_{B(x, s)} |f(y)| \, d\mu(y) \leq \frac{C}{|G_b|} \int_{G_b} \mathcal{M}_d^{b,g} f(x, r) \, dg$$

and hence we obtain (3.2). \[\blacksquare\]

**Proof of Theorem 3.1:** First we prove the implication $(i) \Rightarrow (ii)$. Suppose that there exists $B = B(z, t)$, $0 < t < \infty$ such that

$$\int_B W^{1,p'}(x) \, d\mu(x) = \infty.$$
Then $W^{-1} \chi_B \not\in L^{p'}(W)$ and thus, by Lemma 2.1, there exists a positive function $f \in L^p(W)$ such that

$$\int_B f(x) \, d\mu(x) = \infty.$$ 

Therefore given $(x, r) \in \tilde{X}$, there exists $s \geq r$ such that $B \subset B(x, s)$ and hence $Mf(x, r) = \infty$. Since $\beta$ is a positive measure, we have a contradiction of the condition (i). Thus

$$\int_B W^{1-p'}(x) \, d\mu(x) < \infty.$$ 

To obtain the inequality in (ii) it is sufficient to choose $f(x) = W^{1-p'}(x) \chi_B(x)$ in the hypothesis.

Let us prove (ii)⇒(i). We fix a positive integer $b$, $g \in G$ and $Q \in A^b_k$, $-b \leq k \leq b$. Then, by Lemma 1.1(i) there exist $x_Q \in Q$ such that $Q \subset B(x_Q, \lambda^{k+1})$ and $\mu(B(x_Q, \lambda^{k+1})) \leq C \mu(Q)$. We write $B = B(g^{-1}x_Q, \lambda^{k+1})$, $Q' = g^{-1}Q$ and $\nu = W^{1-p'}$. Since $\nu \in A_\infty(X)$, there exist positive constants $C_{\nu}$ and $\delta$, depending only on $\nu$, such that

$$\frac{\mu(Q')}{\mu(B)} \leq C_{\nu} \left( \frac{\nu(Q')}{\nu(B)} \right)^{\delta}.$$ 

Therefore

$$\nu(B) \leq C_{\nu}^{1/\delta} \left( \frac{\mu(B)}{\mu(Q')} \right)^{1/\delta} \nu(Q') \leq C_1 \nu(Q').$$

Then by the hypothesis and (3.1) we obtain

$$\int_{Q'} \left[ M^b_{d,g} (W^{1-p'} \chi_{Q'})(x, r) \right]^p \, d\beta(x, r) \leq C_2 \int_B \left[ M(\nu \chi_B)(x, r) \right]^p \, d\beta(x, r) \leq C_3 \nu(B) \leq C_4 \int_{Q'} W^{1-p'}(x) \, d\mu(x).$$

Since the constant $C_4$ depends only on $p$, $W$ and $\beta$, then by Theorem 2.1 and Remark 2.1, there exists a constant $C_5$ such that,

(3.3) $$\int_X [M^b_{d,g} f(x, r)]^p \, d\beta(x, r) \leq C_5 \int_X |f(x)|^p W(x) \, d\mu(x)$$

for all $f \in L^p(W)$ and all $g \in G$. Then, it follows by (3.2), (3.3) and by Jensen’s inequality that

$$\int_X [M^b f(x, r)]^p \, d\beta(x, r) \leq \int_X \left( \frac{C_6}{|G_b|} \int_{G_b} M^b_{d,g} f(x, r) \, dg \right)^p \, d\beta(x, r)$$
The result follows by the Monotone Convergence Theorem.

**Remark 3.1.** (a) For $W \equiv 1$, the condition (ii) of Theorem 3.1 is given by

\[
\int_{\tilde{B}} [\mathcal{M}(\chi_B)(x,r)]^p \, d\beta(x,r) \leq C \mu(B)
\]

for all balls $B$. Let us fix $B = B(z,t)$, $0 < t < \infty$. Then, it follows as in the proof of inequality (3.1) of Lemma 3.1 that there exists a constant $C > 0$ such that

\[
C \leq \mathcal{M}(\chi_B)(x,r) \leq 1
\]

for all $(x, r) \in \tilde{B}$. Therefore, from (3.4) we obtain

\[
C^p \beta(\tilde{B}) \leq \int_{\tilde{B}} [\mathcal{M}(\chi_B)(x,r)]^p \, d\beta(x,r) \leq C \mu(B)
\]

Then, the condition (3.4) implies the condition:

\[
\beta(\tilde{B}) \leq C \mu(B)
\]

for a constant $C > 0$ and all balls $B$. But, from the condition (3.5) we obtain

\[
\int_{\tilde{B}} [\mathcal{M}(\chi_B)(x,r)]^p \, d\beta(x,r) \leq \beta(\tilde{B}) \leq C \mu(B),
\]

and therefore the conditions (3.4) and (3.5) are equivalent. The condition (3.5) is the Carleson condition for the homogeneous space $X$ (see [8]).

(b) Let $B = B(z,t)$, $0 < t < \infty$ and $\nu = W^{1-p'}$. Then

\[
C \left( \frac{\nu(B)}{\mu(B)} \right) \leq \mathcal{M}(\nu \chi_B)(x,r)
\]

for all $(x, r) \in \tilde{B}$. Therefore, from the condition (ii) of Theorem 3.1 we obtain

\[
\beta(\tilde{B})^{1/p} = \left( \frac{\mu(B)}{\nu(B)} \right) \left[ \int_{\tilde{B}} \left( \frac{\nu(B)}{\mu(B)} \right)^p \, d\beta(x,r) \right]^{1/p}
\]

\[
\leq C \left( \frac{\mu(B)}{\nu(B)} \right) \left[ \int_{B} [\mathcal{M}(\nu \chi_B)(x,r)]^p \, d\beta(x,r) \right]^{1/p}
\]

\[
\leq C' \left( \frac{\mu(B)}{\nu(B)} \right)^{1/p} \nu(B)^{1/p}.
\]
Then, the condition (ii) of Theorem 3.1 implies the condition:

\[(3.6) \quad \frac{\beta(B)^{1/p}}{\mu(B)} \left( \int_B W^{1-p'}(x) \, d\mu(x) \right)^{1/p'} \leq C\]

for a constant $C > 0$ and all balls $B$. It was proved in Ruiz-Torrea [8] that the condition (3.6) is a necessary and sufficient condition for $M$ to be a bounded operator from $L^p(X, W(x) \, d\mu(x))$ into weak $L^p(\mathbb{X}, \beta)$.

4 – The boundedness of the Poisson integral

Let $\xi : [0, \pi]^{n-1} \times [0, 2\pi] \to S^n$ be the function defined by $\xi(\theta) = \xi(\theta_1, \ldots, \theta_n) = (x_1, \ldots, x_{n+1})$, where

\[
\begin{align*}
  x_1 &= \cos \theta_1; \\
  x_i &= \cos \theta_i \prod_{j=1}^{i-1} \sin \theta_j, \quad 2 \leq i \leq n; \\
  x_{n+1} &= \prod_{j=1}^{n} \sin \theta_j.
\end{align*}
\]

We identify $S^n \times [0, 1]$ with the ball $\mathbb{B} = \{ y \in \mathbb{R}^{n+1} : |y| \leq 1 \}$ using the application $(y, r) \mapsto ry$. If $f$ is a real and integrable function on $S^n$ we define $\mathcal{M}f(y) = \mathcal{M}f(y', h(|y|))$ for $y \in \mathbb{B}$, $y \neq 0$, $y' = y'/|y|$.

In Rauch [6] it was proved that

\[
\begin{align*}
  u_f(y') &= \sup_{0 \leq r < 1} |u_f(ry')| \leq C_n \mathcal{M}f(y'), \\
  y' &\in S^n, \quad f \in L^1(S^n).
\end{align*}
\]

The inequality in the following lemma generalizes the above inequality.

**Lemma 4.1.** There exists a constant $C > 0$ such that, for all real-valued integrable function $f$ on $S^n$ and all $y \in \mathbb{B}$, $0 < |y| < 1$, we have

\[
\begin{align*}
  u_f(y) &\leq C \mathcal{M}f(y) \quad \text{for all } y \in \mathbb{B}.
\end{align*}
\]

**Proof:** We may assume $y = r \mathbb{1} = r(1, 0, \ldots, 0)$, $0 \leq r < 1$. Let us denote $\theta = (\theta_1, \ldots, \theta_n)$, $\theta' = (\theta_2, \ldots, \theta_n)$, $\omega(\theta') = \sin^{n-2} \theta_2 \cdots \sin \theta_{n-1}$ and

\[
(4.1) \quad p(\theta_1, r) = P_r(\xi(\theta)) = \frac{1}{\omega_n} \frac{1 - r^2}{(1 - 2r \cos \theta_1 + r^2)^{(n+1)/2}}.
\]

Then

\[
\begin{align*}
  u_f(r \mathbb{1}) &= \int_0^\pi d\theta_1 \cdots \int_0^{2\pi} p(\theta_1, r) f(\xi(\theta)) \sin^{n-1} \theta_1 \omega(\theta') \, d\theta_n.
\end{align*}
\]
If $0 \leq r \leq 1/2$, we have that $p(\theta_1, r) \leq 2^{n+1}/\omega_n$ and hence

$$|u_f(\approx)| \leq \frac{2^{n+1}}{\omega_n} \int_{S^n} |f(x)| \, d\sigma(x) \leq 2^{n+1} \mathcal{M} f(\approx).$$

Now, let us suppose $1/2 \leq r < 1$. If $m(r) = \arccos r(2 - r)$, then, integrating by parts with respect to $\theta_1$, we obtain

$$I_r = \left| \int_{S^n \setminus B(\approx, h(r))} P_{r\approx}(x) f(x) \, d\sigma(x) \right|$$

$$\leq p(\pi, r) \int_0^\pi d\theta_1 \cdots \int_0^{2\pi} |f(\xi(\theta))| \sin^{n-1} \theta_1 \omega(\theta') \, d\theta_n$$

$$+ p(m(r), r) \int_0^{m(r)} d\theta_1 \int_0^\pi d\theta_2 \cdots \int_0^{2\pi} |f(\xi(\theta))| \sin^{n-1} \theta_1 \omega(\theta') \, d\theta_n$$

$$+ \int_0^\pi \frac{\partial p(\theta_1, r)}{\partial \theta_1} \left( \int_0^{\theta_1} \left( \int_0^\pi d\theta_2 \cdots \int_0^{2\pi} |f(\xi(t, \theta'))| \sin^{n-1} \omega(\theta') \, d\theta_n \right) dt \right) \, d\theta_1$$

$$= I^1_r + I^2_r + I^3_r.$$

We have that

$$I^1_r = \frac{1}{\omega_n} \frac{1-r}{(1+r)^n} \int_{S^n} |f(x)| \, d\sigma(x) \leq \mathcal{M} f(\approx).$$

Since

$$\frac{\omega_{n-1}}{n} 2^{n-1} (1-r)^n \leq \sigma(B(\approx, h(r))) \leq \frac{2^n \omega_{n-1}}{n} (1-r)^n$$

then for $1/2 \leq r < 1$, it follows that

$$I^2_r \leq \frac{2}{\omega_n (1-r)^n} \int_{B(\approx, h(r))} |f(x)| \, d\sigma(x) \leq \frac{2^{n+1} \omega_{n-1}}{n \omega_n} \mathcal{M} f(\approx).$$

Using properties of the Poisson kernel and integration by parts, we obtain

$$\int_0^\pi \frac{\partial p(\theta_1, r)}{\partial \theta_1} \left( \int_0^{\theta_1} \sin^{n-1} t \, dt \right) d\theta_1 = \frac{1}{\omega_{n-1}} \left( \frac{1-r}{(1+r)^n} \right) \leq \frac{1}{\omega_{n-1}}$$

and thus

$$I^3_r \leq \frac{1}{\omega_{n-1}} \mathcal{M} f(\approx).$$

Therefore, there exists a constant $D > 0$, such that

$$I_r \leq I^1_r + I^2_r + I^3_r \leq D \mathcal{M} f(\approx).$$
for all $1/2 \leq r < 1$. Consequently

\[
|u_f(r \mathbb{I})| \leq \frac{2}{\omega_n (1-r)^n} \int_{B(\mathbb{I}, h(r))} |f(x)| \, d\sigma(x) + I_r
\]

\[
\leq \frac{2^{n+1} \omega_{n-1}}{n \omega_n} \frac{1}{\sigma(B(\mathbb{I}, h(r)))} \int_{B(\mathbb{I}, h(r))} |f(x)| \, d\sigma(x) + D\mathcal{M}f(r \mathbb{I})
\]

\[
\leq \frac{2^{n+1} \omega_{n-1}}{n \omega_n} \mathcal{M}f(r \mathbb{I}) + D\mathcal{M}f(r \mathbb{I}) = C\mathcal{M}f(r \mathbb{I}) .
\]

**Proof of Theorem 4.1:** The Proof of (i)$\Rightarrow$(ii) is exactly as the proof of (i)$\Rightarrow$(ii) in Theorem 2.1 and Theorem 3.1.

Let us prove (ii)$\Rightarrow$(i). Let $f$ be a real-valued positive integrable function on $S^n$. There exists a constant $C > 0$, depending only on $n$, such that

\[
P_{ry}(x) \geq \frac{C}{\sigma(B(y', h(r)))}
\]

for all $0 \leq r < 1$, $y' \in S^n$ and $x \in B(y', h(r))$. Therefore

\[
u_f(ry') \geq \frac{C}{\sigma(B(y', h(r)))} \int_{B(y', h(r))} f(x) \, d\sigma(x)
\]

and hence

(4.2) \quad \nu_f^*(ry') \geq C\mathcal{M}f(ry')

Consider the function $k : \mathbb{B} \to \dot{S}^n$ defined by $k(x) = (x/|x|, h(|x|))$, $x \neq 0$, $k(0) = (\mathbb{I}, 0)$, $\mathbb{I} = (1, 0, \ldots, 0)$. Then applying Theorem 3.1 to $X = S^n$ and to the image measure $\beta$ of $\nu$ by $k$, $\beta(A) = \nu(k^{-1}(A))$ and using the inequalities (4.1) and (4.2), we obtain the wanted proof.

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