ON MODULI OF REGULAR SURFACES
WITH \( K^2 = 8 \) AND \( p_g = 4 \)

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Abstract: Let \( S \) be a surface of general type with not birational bicanonical map and that does not contain a pencil of genus 2 curves. If \( K_S^2 = 8 \), \( p_g(S) = 4 \) and \( q(S) = 0 \) then \( S \) can be given as double cover of a quadric surface. We show that its moduli space is generically smooth of dimension 38, and single out an open subset. Note that for these surfaces \( h^2(S, T_S) \) is not zero.

1 – Introduction

It is known that if \( X \) is a surface of general type with a pencil of genus 2 curves then its bicanonical map is non birational (see [1]). On the other hand, there are also surfaces with non birational bicanonical map, which have no pencil of curves of genus 2. According with [2], these surfaces are said to be special. Under the assumption that \( p_g \geq 4 \), the classification of all special surfaces has been completed in [2]. There are three main types of such surfaces, while others of them can be obtained by specialization. Two of these types are classically known (see [1] and [3]), and also their moduli space has been studied (see [4]). Here we concern with the third type, discovered in [2]. These surfaces have the following invariants: \( K^2 = 8 \), \( p_g = 4 \) and \( q = 0 \).

We recall theorem (3.1) in [2].

Theorem 1.1. If \( S \) is a minimal regular surface of general type with \( K^2 = 8 \) and \( p_g = 4 \) which is special, then \( S \) is one of the following types:

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The canonical system $K$ has four distinct simple base points $p_1, p_2, q_1, q_2$. The canonical map $\phi_{K_S}$ is of degree 2 onto a smooth quadric $Q$ of $\mathbb{P}^3$. If $p: \tilde{S} \rightarrow S$ is the blow-up of the points $p_1, p_2, q_1, q_2$, then there exists a morphism $\varphi: \tilde{S} \rightarrow Q \subset \mathbb{P}^3$ such that $\varphi = \phi_{K_S} \circ p$. The morphism $\varphi$ is generically finite of degree 2, with branch curve $B$ on $Q$ of type $B = \eta_1 + \eta_2 + \eta'_1 + \eta'_2 + B'$, where $\eta_1, \eta'_1$ are two distinct lines of the same ruling of $Q$, $\eta_2, \eta'_2$ are two distinct lines of the other ruling, $B'$ is a curve of type $(8,8)$ not containing $\eta_i, \eta'_i$, having 4-uple points at the intersection of the four lines, and no further essential singularity.

The canonical system $|K_S|$ has a fixed component which is an irreducible (-2)-curve $Z$. The linear system $|K_S - Z|$ has no fixed component but has two distinct simple base points. The canonical map $\phi_{K_S}$ has degree 2 onto a smooth quadric $Q$ of $\mathbb{P}^3$. If $p: \tilde{S} \rightarrow S$ is the blow-up of the base points of $|K_S - Z|$, then there exists a morphism $\varphi: \tilde{S} \rightarrow Q$ such that $\varphi = \phi_{K_S} \circ p$. The morphism $\varphi$ is generically finite of degree 2, with branch curve $B$ on $Q$ of type $B = \eta + \eta' + B'$, where $\eta, \eta'$ are two distinct lines of the same ruling of $Q$, $B'$ is a curve of type $(8,8)$ not containing $\eta, \eta'$, having two $[4,4]$-points at the intersection of the $\eta, \eta'$ with a line of the other ruling, and tangent lines $\eta, \eta'$, and no further essential singularity.

Remark 1.2. Here, the essential singularities are the ones that affect the invariants of $S$.

The surfaces in theorem 1.1(ii) are specialization of the surfaces in theorem 1.1(i). We call general the latter surfaces and particular the former ones (see remark (3.10) in [2]).

For the tangent bundle one has

\[ \chi(T_S) = -10 \chi(O_S) + 2 K_S^2 = -34. \]  

We will prove the following:

Theorem 1.3. The family $\mathcal{F}$ of regular surfaces with $K^2 = 8$, $p_g = 4$ with non trivial torsion and without a pencil of genus 2 curves described in theorem 1.1(i) corresponds to an open subset of its moduli space, which is irreducible, smooth of dimension 38.

The prove is based on the geometric description of $S$ by means of the double map on the quadric surface $Q$. 
1.1. Notations and set up

We recall the notations used in [2]: we consider \( n = \eta_1 \cap \eta_2, \, n' = \eta'_1 \cap \eta'_2, \)
\( m = \eta_2 \cap \eta'_1, \, m' = \eta_1 \cap \eta'_2, \) points on the quadric \( Q. \) We denote by \( E_i, E'_i \) for \( i = 1, 2 \) the exceptional curves in \( \tilde{S} \) corresponding to the points \( p_1, p_2, q_1, q_2 \) of \( S \) by the blow up \( p. \)

We introduce further notations. We denote by \( \Gamma_1 \) and \( \Gamma_2 \) the two pencils of lines on \( Q \) to which \( \eta_1 \) and \( \eta_2 \) belong respectively. Let \( bl : Y \to Q \) be the blow up of \( Q \) on \( n, n', m, m', \) and denote by \( E_n, E_{n'}, E_m, E_{m'} \) the exceptional curves corresponding to the points \( n, n', m, m'. \) We write

\[
E = E_n + E_{n'} + E_m + E_{m'}.
\]

We mark with a bar the strict transforms of the divisors of \( Q \) on \( Y. \)

We have the following commutative diagram (cf. the proof of theorem (3.1)(i) in [2]):

\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{p} & S \\
\downarrow \psi & & \downarrow \phi_K \\
Y & \xrightarrow{bl} & Q
\end{array}
\]

The curves \( E_1, E_2, E'_1, E'_2 \) are sent to the lines \( \eta_1, \eta_2, \eta'_1, \eta'_2 \) respectively, by \( \phi_K \circ p. \)

Note that \( \psi : \tilde{S} \to Y \) is a 2:1 morphism branched along a divisor \( B_Y \) of \( Y. \) In fact, there are curves on \( \tilde{S}, \) denoted by \( \tilde{N}, \tilde{N}', \tilde{M}, \tilde{M}' \) in [2], which are sent on \( E_n, E_{n'}, E_m, E_{m'} \) respectively. By theorem 1.1 \( B_Y \) belongs to the linear system

\[
|10 \Gamma_1 + 10 \Gamma_2 - 6 E| = \bar{n}_1 + \bar{n}_2 + \bar{n}'_1 + \bar{n}'_2 + |B'_Y|,
\]

where

\[
B'_Y \in |8 \Gamma_1 + 8 \Gamma_2 - 4 E|.
\]

Since \( Y \) and \( \tilde{S} \) are smooth and \( \psi \) is finite, the branch locus \( B_Y \) is smooth.

2 – The number of moduli of \( S \)

It is possible to compute the number of moduli of the surface \( \tilde{S} \) (and therefore of \( S \)) by applying the projection formula to the tangent sheaf:

\[
h^i(\tilde{S}, T_{\tilde{S}}) = h^i\left(Y, T_Y(-\log B_Y)\right) + h^i\left(Y, T_Y(-D)\right), \quad i = 0, 1, 2,
\]
where \( 2D \sim B_Y \) (cf. [6]). Note that
\[
D \in |5\Gamma_1 + 5\Gamma_2 - 3E|.
\]

**Proposition 2.1.**
\[
h^2\left(Y, T_Y(-\log B_Y)\right) = 0.
\]

**Proof:** Consider the exact sequence
\[
0 \to T_Y(-\log B_Y) \to T_Y \to \mathcal{O}_{B_Y}(B_Y) \to 0.
\]
The curve \( B_Y \) is the disjoint union of 5 components: there are 4 rational curves composing \( E \), plus the curve \( B'_Y \), of genus 43, which can be easily computed by adjunction formula. By Serre duality, \( H^1(B_Y, \mathcal{O}_{B_Y}(B_Y)) = 0 \). Moreover \( H^2(Y, T_Y) = H^2(Q, T_Q) = 0 \). Hence, the long exact sequence of cohomology coming from (4) implies that \( H^2(Y, T_Y(-\log B_Y)) \cong H^2(Y, T_Y) = 0 \).

**Lemma 2.2.**
\[
H^k\left(Y, bl^*T_Q(-D)\right) = 0, \quad \text{for } k = 0, 2,
\]
\[
H^1\left(Y, bl^*T_Q(-D)\right) \cong \mathbb{C}^8.
\]

**Proof:** One has
\[
H^k\left(Y, bl^*T_Q(-D)\right) = H^k\left(Y, \mathcal{O}_Y(-3\Gamma_1 - 5\Gamma_2 - 3E)\right) \oplus H^k\left(Y, \mathcal{O}_Y(-5\Gamma_1 - 3\Gamma_2 - 3E)\right).
\]
In fact \( T_Q = \mathcal{O}_Q(2\Gamma_1) \oplus \mathcal{O}_Q(2\Gamma_2) \). The rest follows from Riemann–Roch formula.

**Proposition 2.3.** \( h^1(S,T_S) \leq 38 \).

**Proof:** Since \( h^0(S,T_S) = 0 \), being \( S \) of general type, then \( h^1(S,T_S) = h^2(S,T_S) - \chi(T_S) = 34 + h^2(S,T_S) \), by (1). The proposition follows once we prove that \( h^2(S,T_S) \leq 4 \). It is sufficient to verify that \( h^2(S,T_S) \leq 4 \). In fact, it is \( h^2(S,T_S) = h^2(S,T_S) \), see for instance [5].

Consider now the exact sequence
\[
0 \to T_Y(-D) \to bl^*T_Q(-D) \to N_{E/Y}^*(-D) \to 0.
\]
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Since $N^*_{E/Y}(-D) = \mathcal{O}_p(-2)^{\oplus 4}$, we get

$$H^0(Y, N^*_{E/Y}(-D)) = 0 \quad \text{and} \quad H^1(Y, N^*_{E/Y}(-D)) = \mathbb{C}^4.$$  

From (5) and lemma 2.2 one has the following exact sequence:

$$0 \to H^1(Y, T_Y(-D)) \to \mathbb{C}^8 \to \mathbb{C}^4 \to H^2(Y, T_Y(-D)) \to 0.$$  

In particular, $h^2(Y, T_Y(-D)) \leq 4$. From (3) and proposition 2.1 one finally has:

$$h^2(\tilde{S}, T_{\tilde{S}}) = h^2(Y, T_Y(-\log B_Y)) + h^2(Y, T_Y(-D)) \leq 4.$$  

2.1. Proof of Theorem 1.3

The irreducibility has been proved in [2].

Consider the family $\mathcal{F}$ of surfaces as in theorem 1.1. It is sufficient to show that $\dim \mathcal{F} = h^1(S, T_S) = 38$. Since the general surface $S$ of $\mathcal{F}$ is the double cover of a nonsingular quadric $Q$ of $\mathbb{P}^3$ branched on a divisor $B$, we can compute the dimension $\dim \mathcal{F}$ by computing the dimension of the linear system $\Sigma(B)$ of the divisors $B$. We recall that $B = \eta_1 + \eta_2 + \eta_1' + \eta_2' + B'$, where $\eta_1 + \eta_1'$ and $\eta_2 + \eta_2'$ are lines of the same pencil on the quadric, $B'$ belongs to the sublinear system $\Sigma(B')$ cut on $Q$ by the surfaces of degree 8, having quadruple points at the 4 intersection points of the 4 lines. Thus $\dim \Sigma(B) = 4 + (\dim \Sigma(B'))$. Since each of the quadruple points gives 10 conditions then

$$\dim \Sigma(B') = h^0(Q, \mathcal{O}_Q(8)) - 40 - 1 = 40.$$  

Hence

$$\dim \mathcal{F} = \dim \Sigma(B) - \dim \text{Aut}(Q) = 40 + 4 - 6 = 38.$$  

Therefore $h^1(S, T_S) \geq \dim \mathcal{F} = 38$. By proposition 2.3, the equality holds. \hfill \blacksquare

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REFERENCES


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