A bijection between noncrossing and nonnesting partitions of types A and B

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Noncrossing and nonnesting set partitions

A set partition of \([n] = \{1, \ldots, n\}\) is a collection of disjoint nonempty subsets of \([n]\), called blocks, whose union is \([n]\).

\[\pi = \{\{1, 3, 4\}, \{2, 6\}, \{5\}\} \text{ is a partition of } [6] \text{ of type } (3, 2, 1)\]

\[\text{op}(\pi) = \{1, 2, 5\}, \quad \text{cl}(\pi) = \{4, 5, 6\}, \quad \text{tr}(\pi) = \{3\}\]

\[m(\pi) = (\text{op}(\pi), \text{cl}(\pi), \text{tr}(\pi))\]
A **complete matching** of $[2n]$ is a set partition of $[2n]$ of type $(2, \ldots, 2)$

A **partial matching** of $[n]$ is a set partition of $[n]$ of type $(2, \ldots, 2, 1, \ldots, 1)$

The triple $m(\pi) = (op(\pi), cl(\pi), tr(\pi))$ encodes some useful information about the set partition $\pi$:

- The number of blocks is $|op(\pi)| = |cl(\pi)|$;
- The number of singleton blocks is $|op(\pi) \cap cl(\pi)|$;
- $\pi$ is a partial matching if and only if $tr(\pi) = \emptyset$;
- $\pi$ is a complete matching if and only if $tr(\pi) = \emptyset$ and $op(\pi) \cap cl(\pi) = \emptyset$. 
Noncrossing set partitions

A set partition $\pi$ of $[n]$ is said **noncrossing** if whenever $a < b < c < d$ are such that $a, c$ are contained in a block $B$ and $b, d$ are contained in a block $B'$ of $\pi$, then $B = B'$.

The set partition $\{\{1, 4, 5, 6\}, \{2, 3\}\}$ is noncrossing:

```
1 2 3 4 5 6
```

while the set partition $\{\{1, 3, 4\}, \{2, 6\}, \{5\}\}$ is not:

```
1 2 3 4 5 6
```
Nonnesting set partitions

A set partition $\pi$ of $[n]$ is said **nonnesting** if whenever $a < b < c < d$ are such that $a, d$ are contained in a block $B$ and $b, c$ are contained in a block $B'$ of $\pi$, then $B = B'$.

The set partition $\{\{1, 3\}, \{2, 4, 5, 6\}\}$ is nonnesting:

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

while the set partition $\{\{1, 4, 5, 6\}, \{2, 3\}\}$ is not:

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]
Absolute order

Let \((W, S)\) be a finite Coxeter system with set of reflections \(T\). Given \(w \in W\), the **absolute length** \(\ell_T(w)\) of \(w\) is the minimal integer \(k\) for which \(w\) can be written as the product of \(k\) reflections:

\[
\ell_T(w) = \min\{ k : w = t_1 \cdots t_k, \text{ for some } t_i \in T \}.
\]

**Definition**

Define the **absolute order** on \(W\) by letting

\[
v \leq_T w \text{ if and only if } \ell_T(w) = \ell_T(v) + \ell_T(v^{-1}w)
\]

for all \(v, w \in W\).
Proposition

Given $w, v \in W$, $v \leq_T w$ if and only if there is a shortest factorization of $w$ as a product of reflections having as a prefix such a shortest factorization for $v$.

$W = S_3$, $S = \{s_1 = (1, 2), s_2 = (2, 3)\}$

$T = \{s_1, s_2, s_1s_2s_1 = (1, 3)\}$
(W, S) finite Coxeter system, with S = \{s_1, \ldots, s_n\}

A **Coxeter element** of W is any element of the form

\[ c = s_{\sigma(1)} \cdots s_{\sigma(n)}, \]

for some permutation \( \sigma \) of the set \([n]\).

**Proposition**

(a) Any two Coxeter elements of W are conjugate.
(b) The Coxeter elements are a subclass of maximal elements in W.
(c) If \( c, c' \) are Coxeter elements, then \([e, c] \cong [e, c']\).
Noncrossing partitions

**Definition**
Let $W$ be a finite reflection group and $c \in W$ a Coxeter element. The poset of noncrossing partitions of $W$ is the interval

$$NC(W) := [e, c] = \{ w \in W : e \leq_T w \leq_T c \}.$$

**Theorem (Reiner, Bessis-Reiner)**
Let $W$ be a finite reflection group. Then,

$$|NC(W)| = Cat(W) := \prod_{i=1}^{n} \frac{d_i + h}{d_i} = \frac{1}{|W|} \prod_{i=1}^{n} (d_i + h),$$

where

(i) $n$ is the number of simple reflections in $W$,
(ii) $h$ is the Coxeter number, and
(iii) $d_1, \ldots, d_n$ are the degrees of the fundamental invariants.
\[ \text{Cat}(W) \] for the finite irreducible Coxeter groups

<table>
<thead>
<tr>
<th>( A_{n-1} )</th>
<th>( B_n )</th>
<th>( D_n )</th>
<th>( l_2(m) )</th>
<th>( H_3 )</th>
<th>( H_4 )</th>
<th>( F_4 )</th>
<th>( E_6 )</th>
<th>( E_7 )</th>
<th>( E_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{n+1} \binom{2n}{n} )</td>
<td>( \binom{2n}{n} )</td>
<td>( \frac{3n-2}{n} \binom{2n-2}{n-1} )</td>
<td>( m + 2 )</td>
<td>32</td>
<td>280</td>
<td>105</td>
<td>833</td>
<td>4160</td>
<td>25080</td>
</tr>
</tbody>
</table>
Noncrossing partitions of type $A_{n-1}$

c = (1, 2, \ldots, n) \text{ Coxeter element}

$\pi \leq_T c$ iff all cycles in $\pi$ are increasing and pairwise noncrossing

$\mathcal{NC}(A_{6-1}) \ni \pi = (1456)(23) \iff \pi = \{\{1, 4, 5, 6\}, \{2, 3\}\} \in \mathcal{NC}([6])$
Noncrossing partitions of type $B_n$

$B_n$ group of sign permutations $\pi$ of $[\pm n] = \{1, 2, \ldots, n, 1, 2, \ldots, n\}$ such that $\pi(i) = \pi(i)$

$p = (\bar{5}, 1, 2)(5, \bar{1}, 2)(3, 4)(\bar{3}, 4) \in B_5$ of type $(3, 2)$

$m(\pi) = (op(\pi) = \{3\}, cl(\pi) = \{2, 4, 5\}, tr(\pi) = \{1\})$

$B_n \leftrightarrow A_{2n-1}$

$i \leftrightarrow i$, if $i \in [n]$  

$i \leftrightarrow n - i$, if $i \in [\bar{1}, \ldots, \bar{n}]$

$NC(B_n)$ is the subset of $NC([\pm n]) = NC([2n])$ consisting of all partitions that are invariant under the map $i \leftrightarrow \bar{i}$
Noncrossing partitions of type $B_n$

$B_n$ group of sign permutations $\pi$ of $[\pm n] = \{\overline{1}, \overline{2}, \ldots, \overline{n}, 1, 2, \ldots, n\}$ such that $\pi(\overline{i}) = \pi(i)$

$\pi = (\overline{5}, 1, 2)(5, \overline{1}, \overline{2})(3, 4)(\overline{3}, \overline{4}) \in B_5$ of type $(3, 2)$

$m(\pi) = (op(\pi) = \{3\}, cl(\pi) = \{2, 4, 5\}, tr(\pi) = \{1\})$

$B_n \hookrightarrow A_{2n-1}$

\[
\begin{align*}
i &\mapsto i, & \text{if } i \in [n] \\
i &\mapsto n - i, & \text{if } i \in [\overline{1}, \ldots, \overline{n}] 
\end{align*}
\]

$NC(B_n)$ is the subset of $NC([\pm n]) = NC([2n])$ consisting of all partitions that are invariant under the map $i \mapsto \overline{i}$
The root poset

Let $W$ be a Weyl group with crystallographic root system $\Phi$, and $\Delta \subseteq \Phi^+$ a set of simple roots

**Definition**

- For $\alpha, \beta \in \Phi^+$, we say that $\alpha \leq \beta$ if and only if $\beta - \alpha \in \mathbb{Z}_{\geq 0}\Delta$. The pair $(\Phi^+, \leq)$ is called the root poset of $W$.

- An antichain in the root poset $(\Phi^+, \leq)$ is called a nonnesting partition of $W$. Let $NN(W)$ denote the set of nonnesting partitions of $W$.

**Theorem**

Let $W$ be a Weyl group. Then,

$$|NC(W)| = |NN(W)| = Cat(W).$$
Nonnesting partitions of type $A_{n-1}$

$\{e_1, \ldots, e_n\}$ canonical basis of $\mathbb{R}^n$

$\Phi = \{e_i - e_j : n \geq i \neq j \geq 1\}$, \hspace{1em} $\Phi^+ = \{e_i - e_j : n \geq i > j \geq 1\}$

$\Delta = \{r_1 = e_2 - e_1, r_2 = e_3 - e_2, \ldots, r_{n-1} = e_n - e_{n-1}\}$

If $i > j$, then $e_i - e_j = r_j + \cdots + r_{i-1} \leftrightarrow (i, j) \in S_n$
Lemma

Let $\alpha = r_i + \cdots + r_j$ and $\beta = r_k + \cdots + r_\ell$ be two roots in $\Phi^+$. Then, \{\alpha, \beta\} is an antichain if and only if $i < k$ and $j < \ell$.

\[ NN(A_4) \ni (r_1, r_2+r_3, r_3+r_4) \leftrightarrow (1, 2)(2, 4)(3, 5) = (1, 2, 4)(3, 5) \in NN([5]) \]

\[ \begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\end{array} \]

- \( supp(r_i + \cdots + r_j) = \{r_i, \ldots, r_j\} \)
- An antichain \((\alpha_1, \ldots, \alpha_k)\) is connected if \( supp(\alpha_i) \cap supp(\alpha_{i+1}) \neq \emptyset \) for \( i = 1, \ldots, k - 1 \)
- The connected components of an antichain \( \pi \) are the connected sub-antichains of \( \pi \) for which the supports of the union of the roots in any two distinct components are disjoint.
Nonnesting partitions of type $B_n$

$\Phi = \{ \pm e_i, 1 \leq i \leq n \} \cup \{ \pm e_i \pm e_j : 1 \leq i \neq j \leq n \}$

$\Phi^+ = \{ e_i : 1 \leq i \leq n \} \cup \{ e_i \pm e_j : 1 \leq j < i \leq n \}$

$\Delta = \{ r_1 = e_1, r_2 = e_2 - e_1, \ldots, r_n = e_n - e_{n-1} \}$

$$e_i = \sum_{k=1}^{i} r_k \leftrightarrow (i, \bar{i})$$

$$e_i - e_j = \sum_{k=j+1}^{i} r_k \leftrightarrow (i, j)(\bar{i}, \bar{j})$$

$$e_i + e_j = 2 \sum_{k=1}^{j} r_k + \sum_{k=j+1}^{i} r_k \leftrightarrow (i, j)(\bar{i}, j)$$
Lemma

- \{r_i + \cdots + r_j, r_k + \cdots + r_\ell\} is an antichain iff \(i < k\) and \(j < \ell\)
- \{2r_1 + \cdots + 2r_i + r_{i+1} + \cdots + r_j, r_k + \cdots + r_\ell\} is an antichain iff \(1 < k\) and \(j < \ell\)
- \{2r_1 + \cdots + 2r_i + r_{i+1} + \cdots + r_j, 2r_1 + \cdots + 2r_k + r_{k+1} + \cdots + r_\ell\} is an antichain iff \(k < i\) and \(j < \ell\)

\[
(2r_1 + 2r_2 + r_3, r_1 + r_2 + r_3 + r_4, r_5) \in NN(B_5)
\]

\[
\downarrow
\]

\[
(2, 3)(\overline{2}, 3)(\overline{5}, 4, 4, 5) \in NN([\pm n])
\]

- \(\text{supp}(2r_1 + 2r_2 + r_3) = \{r_1, r_2, r_3\}\), \(\text{supp}(r_1 + r_2 + r_3 + r_4) = \{r_1, r_2, r_3, r_4\}\), \(\text{supp}(r_5) = \{r_5\}\)

- Connected components: \((2r_1 + 2r_2 + r_3, r_1 + r_2 + r_3 + r_4)\) and \((r_5)\)
The bijection $f : \mathbb{NN}(W) \rightarrow \mathcal{NC}(W)$

$$\pi = (r_1 + r_2, r_2 + r_3, r_3 + r_4 + r_5, r_4 + r_5 + r_6, r_5 + r_6 + r_7)$$

$$= (1, 3, 6)(2, 4, 7)(5, 8) \in \mathbb{NN}(A_7)$$

\[
f(\pi) = (r_1 + \cdots + r_7)f(r_2, r_3, r_4 + r_5, r_5 + r_6)
= (r_1 + \cdots + r_7)r_2r_3f(r_4 + r_5, r_5 + r_6)
= (r_1 + \cdots + r_7)r_2r_3(r_4 + r_5 + r_6)r_5
= (1, 8)(2, 3, 4, 7)(5, 6) \in \mathcal{NC}(A_7), \quad m(\pi) = m(f(\pi))
\]
The bijection $f : \mathcal{NN}(W) \rightarrow \mathcal{NC}(W)$

$\pi = (r_1 + r_2, r_2 + r_3, r_3 + r_4 + r_5, r_4 + r_5 + r_6, r_5 + r_6 + r_7)$

$= (1, 3, 6)(2, 4, 7)(5, 8) \in \mathcal{NN}(A_7)$

\[
f(\pi) = (r_1 + \cdots + r_7) f(r_2, r_3, r_4 + r_5, r_5 + r_6)
= (r_1 + \cdots + r_7) r_2 r_3 f(r_4 + r_5, r_5 + r_6)
= (r_1 + \cdots + r_7) r_2 r_3 (r_4 + r_5 + r_6) r_5
= (1, 8)(2, 3, 4, 7)(5, 6) \in \mathcal{NC}(A_7), \quad m(\pi) = m(f(\pi))
\]

$F_{st} = (1 < 2 < 3 < 4 < 5), \quad L_{st} = (2 < 3 < 5 < 6 < 7)$
\[ \pi = \left( r_1 + r_2, r_2 + r_3, r_3 + r_4 + r_5, r_4 + r_5 + r_6, r_5 + r_6 + r_7 \right) \\
= (1, 3, 6)(1, 3, 6)(4, 7)(4, 7)(5, 2, 2, 5) \in NN(B_7) \]

\[ f(\pi) = \left( r_1 + \cdots + r_7 \right)f \left( r_2, r_3, r_4 + r_5, r_5 + r_6 \right) \\
= (r_1 + \cdots + r_7)r_2r_3f(r_4 + r_5, r_5 + r_6) \\
= (r_1 + \cdots + r_7)r_2r_3(r_4 + r_5 + r_6)r_5 \\
= (7, 7)(1, 2)(1, 2)(2, 3)(2, 3)(3, 6)(3, 6)(4, 5)(4, 5) \in NC(B_7) \]

\[ m(\pi) = m(f(\pi)) = (\{1, 4\}, \{5, 6, 7\}, \{2, 3\}) \]
\[ \pi = (1, 3, 6)(2, 4, 7)(5, 8) \]

\[ f(\pi) = (1, 8)(2, 3, 4, 7)(5, 6) \]
The bijection $f : \mathcal{NN}(W) \rightarrow \mathcal{NC}(W)$

$$\pi = (2r_1 + 2r_2 + 2r_3 + r_4, \ 2r_1 + 2r_2 + r_3 + r_4 + r_5, \ r_3 + r_4 + r_5 + r_6, \ r_4 + r_5 + r_6 + r_7, \ r_6 + r_7 + r_8)$$
The bijection $f : NN(W) \rightarrow NC(W)$

$$\pi = (2r_1 + 2r_2 + 2r_3 + r_4, \ 2r_1 + 2r_2 + r_3 + r_4 + r_5, \ r_3 + r_4 + r_5 + r_6, \ r_4 + r_5 + r_6 + r_7, \ r_6 + r_7 + r_8)$$
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The bijection $f : NN(W) \rightarrow NC(W)$

$$\pi = (2r_1 + 2r_2 + 2r_3 + r_4, 2r_1 + 2r_2 + r_3 + r_4 + r_5, r_3 + r_4 + r_5 + r_6, r_4 + r_5 + r_6 + r_7, r_6 + r_7 + r_8)$$
The bijection $f : \mathcal{NN}(W) \rightarrow \mathcal{NC}(W)$

$\pi = (2r_1 + 2r_2 + 2r_3 + r_4, 2r_1 + 2r_2 + r_3 + r_4 + r_5, r_3 + r_4 + r_5 + r_6, r_4 + r_5 + r_6 + r_7, r_6 + r_7 + r_8)$

$f(\pi) : (2r_1 + 2r_2 + r_3 + \cdots + r_8, 2r_1 + 2r_2 + 2r_3 + 2r_4 + 2r_5 + r_6 + r_7, r_3, f(r_4, r_6))$

$\rightarrow (2r_1 + 2r_2 + r_3 + \cdots + r_8, 2r_1 + 2r_2 + 2r_3 + 2r_4 + 2r_5 + r_6 + r_7, r_3, r_4, r_6)$

$\rightarrow (2, 8)(2, 3, 5, 7)(5, 7)(2, 3)(2, 3)(3, 4)(3, 4)(5, 6)(5, 6)$

$\rightarrow (2, 3, 4, 8)(2, 3, 4, 8)(5, 6, 7)(5, 6, 7) = f(\pi)$
\( f(\pi) = (2, 3, 4, 8)(\bar{2}, \bar{3}, \bar{4}, 8)(5, 6, 7)(\bar{5}, \bar{6}, 7) \)

\[ \rightarrow (2, \bar{8})(\bar{2}, 8)(5, \bar{7})(\bar{5}, 7)(2, 3)(\bar{2}, \bar{3})(3, 4)\bar{3}, \bar{4})(5, 6)(\bar{5}, \bar{6}) \]

\[ \rightarrow (2r_1 + 2r_2 + r_3 + \cdots + r_8, \ 2r_1 + 2r_2 + 2r_3 + 2r_4 + 2r_5 + r_6 + r_7, \ r_3, r_4, r_6) \]

\( D = (3 > 2), \quad F_{st} = (3 < 4 < 6) \)

\( L_{st} = (4 < 5 < 6 < 7 < 8) \)
\[ f(\pi) = (2, 3, 4, \overline{8})(\overline{2}, \overline{3}, \overline{4}, 8)(5, 6, \overline{7})(\overline{5}, \overline{6}, 7) \]

\[ \rightarrow (2, \overline{8})(\overline{2}, 8)(5, \overline{7})(\overline{5}, \overline{7})(2, 3)(\overline{2}, \overline{3})(3, 4)(\overline{3}, \overline{4})(5, 6)(5, \overline{6}) \]

\[ \rightarrow (2r_1 + 2r_2 + r_3 + \cdots + r_8, \ 2r_1 + 2r_2 + 2r_3 + 2r_4 + 2r_5 + r_6 + r_7, \ r_3, r_4, r_6) \]

\[ D = (3 > 2), \quad F_{st} = (3 < 4 < 6) \]

\[ L_{st} = (4 < 5 < 6 < 7 < 8) \]

Then

\[ \pi = (2r_1 + 2r_2 + 2r_3 + r_4, \ 2r_1 + 2r_2 + r_3 + r_4 + r_5, \ r_3 + r_4 + r_5 + r_6, \ r_4 + r_5 + r_6 + r_7, \ r_6 + r_7 + r_8) \]

with \( m(\pi) = m(f(\pi)) = (\{1\}, \{1, 4, 6, 7, 8\}, \{2, 3, 5\}) \)
Theorem

The map $f$ is a bijection between the sets $NN(\Psi)$ and $NC(\Psi)$, for $\Psi = A_{n-1}$ or $\Psi = B_n$, that preserves the number of blocks and the triples $(op(\pi), cl(\pi), tr(\pi))$. 
**Theorem**
The map $f$ is a bijection between the sets $NN(\Psi)$ and $NC(\Psi)$, for $\Psi = A_{n-1}$ or $\Psi = B_n$, that preserves the number of blocks and the triples $(op(\pi), cl(\pi), tr(\pi))$.

**Corollary**
The map $f$ establishes a bijection between nonnesting matching partitions of $[2n]$ and noncrossing matching partitions of $[2n]$. 