A Knéser-Type Theorem for the Equation
\[ x^{(m)} = f(t, x) \]
in Locally Convex Spaces

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**Abstract.** We shall give sufficient conditions for the existence of solutions of the Cauchy problem for the equation \( x^{(m)} = f(t, x) \). We also prove that the set of these solutions is a continuum.

**Keywords:** Differential equations, set of solutions, measures of non-compactness

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Let \( E \) be a quasicomplete locally convex topological vector space, and let \( P \) be a family of continuous seminorms generating the topology of \( E \). Assume that \( I = [0, a] \) and \( B = \{ x \in E : p_k(x) \leq b \ (i = 1, \ldots, k) \} \), where \( p_1, \ldots, p_k \in P \).

In this paper we investigate the existence of solutions and the structure of the set of solutions of the Cauchy problem

\[
\begin{align*}
  x^{(m)} &= f(t, x) \\
  x(0) &= 0 \\
  x'(0) &= \eta_1 \\
  & \vdots \\
  x^{(m-1)}(0) &= \eta_{m-1}
\end{align*}
\] (1)

where \( m \) is a positive integer, \( \eta_1, \eta_2, \ldots, \eta_{m-1} \in E \) and \( f \) is a bounded continuous function from \( I \times B \) into \( E \). Our considerations are a continuation of Szukała’s paper [8]. For other results concerning differential equations in locally convex spaces see [4].

Put

\[ M = \sup \left\{ p_i(f(t, x)) : t \in I, x \in B, i = 1, \ldots, k \right\}. \]

Choose a positive number \( d \) such that \( d \leq a \) and

\[
\sum_{j=1}^{m-1} p_k(\eta_j) \frac{d^j}{j!} + M \frac{d^m}{m!} \leq b \quad (i = 1, \ldots, k). \] (2)
Let $J = [0, d]$. Denote by $C = C(J, E)$ the space of all continuous functions from $J$ into $E$ endowed with the topology of uniform convergence.

For any bounded subset $A$ of $E$ and $p \in P$ we denote by $\beta_p(A)$ the infimum of all $\varepsilon > 0$ for which there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of $E$ such that $A \subset \{x_1, x_2, \ldots, x_n\} + B_p(\varepsilon)$, where $B_p(\varepsilon) = \{x \in E : p(x) \leq \varepsilon\}$. The family $(\beta_p(A))_{p \in P}$ is called the measure of non-compactness of $A$. It is known [6] that:

1° $X$ is relatively compact in $E$ $\iff$ $\beta_p(X) = 0$ for every $p \in P$.
2° $X \subset Y$ $\implies$ $\beta_p(X) \leq \beta_p(Y)$.
3° $\beta_p(X \cup Y) = \max\{\beta_p(X), \beta_p(Y)\}$.
4° $\beta_p(X + Y) \leq \beta_p(X) + \beta_p(Y)$.
5° $\beta_p(\lambda X) = |\lambda|\beta_p(X)$ ($\lambda \in \mathbb{R}$).
6° $\beta_p(X) = \beta_p(\tilde{X})$.
7° $\beta_p(\text{conv } X) = \beta_p(X)$.
8° $\beta_p(\cup_{0 \leq \lambda \leq h} \lambda X) = h\beta_p(X)$.

The following lemma is given in [8].

**Lemma 1.** Let $H$ be a bounded countable subset of $C$. For each $t \in J$ put $H(t) = \{u(t) : u \in H\}$. If the space $E$ is separable, then for each $p \in P$ the function $t \mapsto \beta_p(H(t))$ is integrable and

$$\beta_p \left( \left\{ \int_J u(s) \, ds : u \in H \right\} \right) \leq \int_J \beta_p(H(s)) \, ds.$$ 

Moreover, let us recall the following lemma from [9].

**Lemma 2.** Let $w : [0, 2b] \to \mathbb{R}_+$ be a continuous non-decreasing function and let $g : [0, c) \to [0, 2b]$ be a $C^m$-function satisfying the inequalities

$$g^{(j)}(t) \geq 0 \quad (j = 0, 1, \ldots, m)$$
$$g^{(j)}(0) = 0 \quad (j = 0, 1, \ldots, m - 1)$$
$$g^{(m)}(t) \leq w(g(t)) \quad (t \in [0, c)).$$

If $w(0) = 0$, $w(r) > 0$ for $r > 0$ and $\int_{0^+} (r^{m-1}w(r))^{-\frac{1}{m}} \, dr = \infty$, then $g = 0$.

We can now formulate our main result.

**Theorem.** Suppose that for each $p \in P$ there exists a continuous non-decreasing function $w_p : \mathbb{R}_+ \to \mathbb{R}_+$ such that $w_p(0) = 0$, $w_p(r) > 0$ for $r > 0$ and

$$\int_{0^+} \frac{dr}{\sqrt{r^{m-1}w_p(r)}} = \infty. \quad (3)$$

If

$$\beta_p(f(t, X)) \leq w_p(\beta_p(X)) \quad (4)$$
for \( p \in P, t \in I \) and bounded subsets \( X \) of \( E \), then the set \( S \) of all solutions of problem (1) defined on \( J \) is non-empty, compact and connected in \( C(J, E) \).

**Proof.** 1° Put

\[
    r(x) = \begin{cases} 
        x \left( \frac{1}{K(x)} \right) & \text{for } x \in B \\
        x & \text{for } x \in E \setminus B
    \end{cases}
\]

and \( g(t, x) = f(t, r(x)) \) for \((t, x) \in J \times E\), where \( K \) is the Minkowski functional of \( B \). As \( B \) is a closed, balanced and convex neighbourhood of 0, from known properties of the Minkowski functional it follows that \( r \) is a continuous function from \( E \) into \( B \) and

\[
r(X) \subseteq \bigcup_{0 \leq \lambda \leq 1} \lambda X \quad \text{for any subset } X \text{ of } E.
\]

Thus \( \beta_p(r(X)) \leq \beta_p(X) \) for any \( p \in P \) and any bounded subset \( X \) of \( E \). Consequently, \( g \) is a bounded continuous function from \( J \times E \) into \( E \) such that

\[
    \beta_p(g(t, X)) \leq w_p(\beta_p(X)) \quad (4')
\]

for \( p \in P, t \in J \) and bounded subsets \( X \) of \( E \) and

\[
    p_i(g(t, x)) \leq M \quad (i = 1, \ldots, k; t \in J, x \in E). \quad (5)
\]

We introduce a mapping \( F \) defined by

\[
    F(x)(t) = q(t) + \frac{1}{(m-1)!} \int_0^t (t - s)^{m-1} g(s, x(s)) \, ds \quad (t \in J, x \in C)
\]

where \( q(t) = \sum_{j=1}^{m-1} \eta_j t^j \). It is known (cf. [2]) that \( F \) is a continuous mapping \( C \to C \) and the set \( F(C) \) is bounded and equicontinuous. It is clear from (1) and (5) that if \( x = F(x) \), then

\[
    p_i(x(t)) \leq \sum_{j=1}^{m-1} p_i(\eta_j) \frac{d^j}{j!} + \frac{1}{(m-1)!} \int_0^t (t - s)^{m-1} M \, ds
\]

\[
    \leq \sum_{j=1}^{m-1} p_i(\eta_j) \frac{d^j}{j!} + M \frac{d^m}{m!} \quad (i = 1, \ldots, k)
\]

\[
    \leq b
\]

so \( x(t) \in B \) for \( t \in J \). Therefore, a function \( x \in C \) is a solution of problem (1) if and only if \( x = F(x) \).

2° For any \( n \in \mathbb{N} \) put

\[
    u_n(t) = \begin{cases} 
        0 & \text{if } 0 \leq t \leq \frac{d}{n} \\
        q(t - \frac{d}{n}) + \frac{1}{(m-1)!} \int_0^{t - \frac{d}{n}} (t - s)^{m-1} g(s, u_n(s)) \, ds & \text{if } \frac{d}{n} < t \leq d.
    \end{cases}
\]
Then \( u_n \) is a continuous function \( J \rightarrow B \) and
\[
\lim_{n \rightarrow \infty} (u_n(t) - F(u_n)(t)) = 0
\] (6)
uniformly for \( t \in J \). Let \( V = \{u_n : n \in \mathbb{N}\} \). From (6) it follows that the set \( \{u_n - F(u_n) : n \in \mathbb{N}\} \) is relatively compact in \( C \). Since
\[
V \subseteq \{u_n - F(u_n) : n \in \mathbb{N}\} + F(V)
\] (7)
and the set \( F(V) \) is bounded and equicontinuous, we conclude that the set \( V \) is also bounded and equicontinuous. Hence for each \( p \in P \) the function \( t \mapsto \beta_p(V(t)) \) is continuous on \( J \). Denote by \( H \) a closed separable subspace of \( E \) such that
\[
g(s, u_n(s)) \in H \quad (s \in J, n \in \mathbb{N}).
\]
Let \( \{\beta^H_p\}_{p \in P} \) be the measure of non-compactness in \( H \). Fix \( t \in J \) and \( p \in P \). From (4) we have
\[
\beta^H_p (g(s, V(s))) \leq 2\beta_p (g(s, V(s))) \leq 2w_p (\beta_p(V(s))) \quad (s \in [0, t]).
\]
By Lemma 1, we get
\[
\beta_p(F(V)(t)) = \beta_p \left( \left\{ \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} g(s, u_n(s)) \, ds : n \in \mathbb{N} \right\} \right)
\leq \beta^H_p \left( \left\{ \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} g(s, u_n(s)) \, ds : n \in \mathbb{N} \right\} \right)
\leq \frac{1}{(m-1)!} \int_0^t \beta^H_p \left( \left\{ (t-s)^{m-1} g(s, u_n(s)) : n \in \mathbb{N} \right\} \right) \, ds
= \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \beta^H_p (g(s, V(s))) \, ds
\leq \frac{2}{(m-1)!} \int_0^t (t-s)^{m-1} w_p (\beta_p(V(s))) \, ds.
\]
On the other hand, from (6) and (7) we obtain
\[
\beta_p(V(t)) \leq \beta_p(F(V)(t))
\]
Hence
\[
\beta_p(V(t)) \leq \frac{2}{(m-1)!} \int_0^t (t-s)^{m-1} w_p (\beta_p(V(s))) \, ds \quad (t \in J, p \in P).
Putting
\[ g(t) = \frac{2}{(m - 1)!} \int_0^t (t - s)^{m-1} w_p(\beta_p(V(s))) \, ds \]
we see that
\begin{align*}
&g \in C^m \\
&\beta_p(V(t)) \leq g(t) \\
&g^{(j)}(t) \geq 0 \text{ for } j = 0, 1, \ldots, m \\
&g^{(j)}(0) = 0 \text{ for } j = 0, 1, \ldots, m - 1 \\
&g^{(m)}(t) = 2w_p(\beta_p(V(t))) \leq 2w_p(g(t)) \text{ for } t \in J.
\end{align*}
Moreover, by (3),
\[ \int_{0+} \frac{dr}{\sqrt{r^{m-1}2w_p(r)}} = \infty. \]
By Lemma 2 from this we deduce that \( g(t) = 0 \) for \( t \in J \). Thus \( \beta_p(V(t)) = 0 \) for \( t \in J \) and \( p \in P \). Therefore for each \( t \in J \) the set \( V(t) \) is relatively compact in \( E \). As the set \( V \) is equicontinuous, Ascoli’s theorem proves that \( V \) is relatively compact in \( C \). Hence the sequence \((u_n)\) has a limit point \( u \). As \( F \) is continuous from (6) we conclude that \( u = F(u) \), i.e. \( u \) is a solution of problem (1). This proves that the set \( S \) is non-empty.

3° Let us first remark that the set \( S \) is compact in \( C \). Indeed, as \((I - F)(S) = \{0\}\), in the same way as in Step 2°, we can prove that \( S \) is relatively compact in \( C \). Moreover, from the continuity of \( F \) it follows that \( S \) is closed in \( C \). Suppose that \( S \) is not connected. Thus there exist non-empty closed sets \( S_0 \) and \( S_1 \) such that \( S = S_0 \cup S_1 \) and \( S_0 \cap S_1 = \emptyset \). As \( S_0 \) and \( S_1 \) are compact subsets of \( C \) and \( C \) is a Tichonov space, this implies (see [3: §41, II, Remark 3]) the existence of a continuous function \( v : C \to [0,1] \) such that \( v(x) = 0 \) for \( x \in S_0 \) and \( v(x) = 1 \) for \( x \in S_1 \). Further, for any \( n \in \mathbb{N} \) we define a mapping \( F_n \) by
\[ F_n(x)(t) = F(x)(r_n(t)) \quad (x \in C, t \in J) \]
where
\[ r_n(t) = \begin{cases} 
0 & \text{for } 0 \leq t \leq \frac{d}{n} \\
\frac{t - d}{n} & \text{for } \frac{d}{n} \leq t \leq d.
\end{cases} \]
It can be easily verified (cf. [10]) that:
\begin{enumerate}
\item \( F_n \) is a continuous mapping \( C \to C \).
\item \( \lim_{n \to \infty} F_n(x) = F(x) \) uniformly for \( x \in C \).
\item \( I - F_n \) is a homeomorphism \( C \to C \) (I - identity mapping).
\end{enumerate}
Fix \( u_0 \in S_0, u_1 \in S_1 \) and \( n \in \mathbb{N} \). Put
\[ e_n(\lambda) = \lambda(u_1 - F_n(u_1)) + (1 - \lambda)(u_0 - F_n(u_0)) \quad (0 \leq \lambda \leq 1). \]
Let \( u_{n\lambda} = (I - F_n)^{-1}(e_n(\lambda)) \). As \( e_n(\lambda) \) depends continuously on \( \lambda \) and \( I - F_n \) is a homeomorphism, we see that the mapping \( \lambda \mapsto u(u_{n\lambda}) \) is continuous on \([0,1]\). Moreover,
$u_{n0} = u_0$ and $u_{n1} = u_1$, so that $v(u_{n0}) = 0$ and $v(u_{n1}) = 1$. Thus there exists $\lambda_n \in [0, 1]$ such that

$$v(u_{n\lambda_n}) = \frac{1}{2}. \quad (8)$$

For simplicity put $v_n = u_{n\lambda_n}$ and $V = \{v_n : n \geq 1\}$. Since $\lim_{n \to \infty} e_n(\lambda) = 0$ uniformly for $\lambda \in [0, 1]$, we get

$$\lim_{n \to \infty} (v_n - F(v_n)) = \lim_{n \to \infty} (e_n(\lambda) + F_n(v_n) - F(v_n)) = 0 \quad (9)$$

and therefore the set $(I-F)(V)$ is relatively compact in $C$. Using now a similar argument as in Step 2°, we can prove that the set $V$ is relatively compact in $C$. Consequently, the sequence $(v_n)$ has a limit point $z$. In view of (9) and the continuity of $F$, we infer that $z \in S$, so $v(z) = 0$ or $v(z) = 1$. On the other hand, from (8) it is clear that $v(z) = \frac{1}{2}$, which yields a contradiction. Thus $S$ is connected.

References


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