Fourth International Conference on
Geometry, Integrability and Quantization
June 6–15, 2002, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2003, pp 296–302

EXACTLY SOLVABLE PERIODIC DARBOUX q-CHAINS

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Abstract. A difference q-analogue of the dressing chain is considered in this paper. The relation between the discrete and continuous models is also discussed.

1. Introduction

Let $L_1, L_2, \ldots$ be self-adjoint differential operators acting on $\mathbb{R}$. They form a **Darboux chain** if they satisfy the relation

$$L_j = A_j A_j^+ - \alpha_j = A_{j-1}^+ A_{j-1} \tag{1}$$

where $A_j = -\frac{\mathrm{d}}{\mathrm{d}x} + f_j(x)$ are first order differential operators. A Darboux chain is called **periodic** if $L_{j+r} = L_j$ for some r and for all $j = 1, 2, \ldots$. The number r is called period of a Darboux chain. In the particular case $r = 1$ the operator $L_1 + \frac{\alpha}{2}$ appears to be the harmonic oscillator and it is known that it has a discrete spectrum consisting of the geometric sequence $\lambda_k = \dfrac{2k+1}{2}\alpha$, where $k = 1, 2 \ldots$. Eigenfunctions of the harmonic oscillator are expressed in terms of the Hermite polynomials and therefore they form a complete family in the Hilbert space $\mathcal{L}_2(\mathbb{R})$.

Periodic Darboux chain leads to the following integrable system of differential equations:

$$(f_j + f_{j-1})' = f_j^2 - f_{j-1}^2 - \alpha_j\,, \qquad j = 1, 2, \ldots \tag{2}$$

where $f_{j+r} \equiv f_j$. Sometimes this system is referred to as **dressing chain** which has been thoroughly examined in [8]. The cases $\alpha = 0$ and $\alpha \neq 0$, where $\alpha = \sum_{j=1}^{r} \alpha_j$, are cardinally different. The operators of a periodic