BOSE-FERMI MIXTURES IN TWO OPTICAL LATTICES

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Abstract. We present stationary and travelling wave solutions for equations describing Bose-Fermi mixtures in an external potentials which are elliptic functions of modulus $k$. There are indications that such waves and localized objects may be observed in experiments with cold quantum degenerate gases.

1. Introduction

Recently, there has been a strong interest on quantum degenerate mixtures of bosons and fermions [3, 14, 16]. In this paper, we study a system of coupled non-linear Schrödinger equations modelling a quantum degenerate mixture of bosons and fermions in optical lattice. Here we extend the results of our recent paper [10] and obtain new exact solutions in elliptic functions for the case when the boson and fermion ingredients are trapped by potentials with different strengths $V_{0,F} \neq V_{0,B}$.

2. Bose-Einstein Mixtures in Optical Lattice: Basic Equations in Mean Field Approximations

In this section we consider a mixture of BEC consisting of one boson and $N_f$ fermion ingredients. In the one-dimensional approximation it is described by the following $N_f + 1$ coupled equations (see [16] and the references therein)

\[ i\hbar \frac{\partial \Psi^b}{\partial t} + \frac{1}{2m_B} \frac{\partial^2 \Psi^b}{\partial x^2} - V_B \Psi^b - g_{BB} |\Psi^b|^2 \Psi^b - g_{BF} \rho_f \Psi^b = 0 \]  

(1)

\[ i\hbar \frac{\partial \Psi^f_j}{\partial t} + \frac{1}{2m_F} \frac{\partial^2 \Psi^f_j}{\partial x^2} - V_F \Psi^f_j - g_{BF} |\Psi^b|^2 \Psi^f_j = 0 \]  

(2)
where $\rho_f = \sum_{i=1}^{N_f} |\Psi^f_i|^2$ and

$$g_{BB} = \frac{2a_{BB}}{a_s}, \quad g_{BF} = \frac{2a_{BF}}{a_s\alpha}, \quad a_s = \sqrt{\frac{\hbar}{m_B \omega_\perp}}. \quad (3)$$
a_{BB} and $a_{BF}$ are the scattering lengths for $s$-wave collisions for boson-boson and boson-fermion interactions, respectively. An appropriate class of periodic potentials to model the quasi-1D confinement produced by a standing light wave is given by [4]

$$V_B = V_{0,B} \text{sn}^2(\alpha x, k), \quad V_F = V_{0,F} \text{sn}^2(\alpha x, k) \quad (4)$$

where $\text{sn}(\alpha x, k)$ denotes the Jacobian elliptic sine function [2] with elliptic modulus $0 \leq k \leq 1$.

Experimental realization of two-component Bose-Einstein condensates have stimulated considerable attention in the quasi-1D regime [7] when the Gross-Pitaevskii equations for two interacting Bose-Einstein condensates reduce to coupled nonlinear Schrödinger (CNLS) equations with an external potential. In specific cases the two component CNLS equations [1, 9, 13] can be reduced to the Manakov system [12] with an external potential. Elliptic solutions for the CNLS and Manakov system were derived in [6, 8, 15].

In the presence of external elliptic potential explicit stationary solutions for NLS were derived in [4, 5]. These results were generalized to the $n$-component CNLS in [7]. For two component CNLS explicit stationary solutions are derived in [11].

### 3. Type A Travelling Wave Solutions with Non-Trivial Phases

At first we restrict our attention to stationary solutions of these CNLS

$$\Psi^b(x, t) = q_0(x) e^{-i \frac{\omega_0}{2} t + i \Theta_0(x) + i \kappa_0} \quad (5)$$

$$\Psi^f_j(x, t) = q_j(x) e^{-i \frac{\omega_j}{2} t + i \Theta_j(x) + i \kappa_{0,j}} \quad (6)$$

where $j = 1, \ldots, N_f$, $\kappa_0$, $\kappa_{0,j}$ are constant phases, $q_j$ and $\Theta_0$, $\Theta_j(x)$ are real-valued functions connected by the relation

$$\Theta_0(x) = C_0 \int_0^x \frac{dx'}{q_0^3(x')}, \quad \Theta_j(x) = C_j \int_0^x \frac{dx'}{q_j^3(x')} \quad (7)$$

$C_0, C_j, j = 1, \ldots, N_f$ being constants of integration. Substituting the Ansatz (5), (6) in Equation (1) and separating the real and imaginary part we get

$$\frac{1}{2m_B} q_{0xx} - g_{BB} q_0^3 - V_B q_0 - g_{BF} \left( \sum_{i=1}^{N_f} q_i^2 \right) q_0 + \omega_0 q_0 = \frac{1}{2m_B} C_0^2 q_0^3 \quad (8)$$
We seek solutions for $q_0^2$ and $q_j^2$, $j = 1, \ldots, N_f$ as a quadratic function of $\text{sn}(\alpha x, k)$

$$q_0^2 = A_0 \text{sn}^2(\alpha x, k) + B_0, \quad q_j^2 = A_j \text{sn}^2(\alpha x, k) + B_j.$$  \hspace{1cm} (10)$$

Equating the coefficients of equal powers of $\text{sn}(\alpha x, k)$ results in the following relations among the solution parameters $\omega, C_j, A_j$ and $B_j$ and the characteristic of the optical lattice $V_0, \alpha$ and $k$:

$$\sum_{j=1}^{N_f} A_j = \frac{\alpha^2 k^2}{g_B} \left( \frac{1}{m_B} - \frac{g_B}{m_F g_B} \right) - \frac{1}{g_B} \left( V_{0B} - V_{0F} g_B \right)$$  \hspace{1cm} (11)$$

$$\omega_0 = \frac{\alpha^2 (k^2 + 1)}{2 m_B} + g_B B_0 + g_B \sum_{i=1}^{N_f} B_i + \frac{\alpha^2 k^2 B_0}{2 m_B A_0} \omega_j = \frac{\alpha^2 (k^2 + 1)}{2 m_B} + g_B B_0 + \frac{\alpha^2 k^2 B_j}{2 m_F A_j}, \quad A_0 = \frac{\alpha^2 k^2 - m_F V_{0F}}{m_F g_B}$$  \hspace{1cm} (12)$$

$$C_0^2 = \frac{\alpha^2 B_0}{A_0} (A_0 + B_0)(A_0 + B_0 k^2), \quad C_j^2 = \frac{\alpha^2 B_j}{A_j} (A_j + B_j)(A_j + B_j k^2)$$  \hspace{1cm} (13)$$

where $j = 1, \ldots, N_f$. Next for convenience we introduce

$$B_0 = -\beta_0 A_0, \quad B_j = -\beta_j A_j, \quad j = 1, \ldots, N_f$$  \hspace{1cm} (14)$$

then

$$C_0^2 = \alpha^2 A_0^2 \beta_0 (\beta_0 - 1)(1 - \beta_0 k^2)$$  \hspace{1cm} (15)$$

$$C_j^2 = \alpha^2 A_j^2 \beta_j (\beta_j - 1)(1 - \beta_j k^2).$$  \hspace{1cm} (16)$$

In order that our results (10) are consistent with the parametrization (5), (6), (7) we must ensure that both $q_0(x)$ and $\Theta_0(x)$ are real-valued, and also $q_j(x)$ and $\Theta_j(x)$ are real-valued; this means that $C_0^2 \geq 0$ and $q_0^2(x) \geq 0$ and also $C_j^2 \geq 0$ and $q_j^2(x) \geq 0$. An elementary analysis shows that one of the following conditions

a) $A_l \geq 0, \quad \beta_l \leq 0$ b) $A_l \leq 0, \quad 1 \leq \beta_l \leq \frac{1}{k^2}$  \hspace{1cm} (17)$$

for $l = 0, \ldots, N_f$ must hold. Using the well known transformation $x \to x - c_j t$, $j = 0, \ldots, N_f$ it is easy to obtain travelling wave solutions with different velocities $c_j$

$$\Psi^b(x, t) = q_0(x - c_0 t) e^{-\frac{2 \pi i}{k^2} t - i \hbar m_B (\frac{i}{2} c_0^2 t + c_0 x) + i \Theta_0(x) + i \kappa_0}$$

$$\Psi^f_j(x, t) = q_j(x - c_j t) e^{-\frac{2 \pi i}{k^2} t - i \hbar m_B (\frac{i}{2} c_j^2 t + c_j x) + i \Theta_j(x) + i \kappa_{0,j}}$$
where \( j = 1, \ldots, N_f \), \( \kappa_0, \kappa_{0,j} \), are constant phases, \( q_j \) and \( \Theta_0, \Theta_j(x) \) are real-valued functions connected by the relation
\[
\Theta_0(x, t) = C_0 \int_0^{x-\omega_0 t} \frac{dx'}{q_0^2(x')}, \quad \Theta_j(x, t) = C_j \int_0^{x-\omega_j t} \frac{dx'}{q_j^2(x')}, \quad j = 1, \ldots, N_f.
\]

We display also mixed type solutions for which the boson part has trivial phase while the fermions have nontrivial phases and vice versa. These are obtained with

1. generic \( C_j \) and \( B_0 = -A_0 \), \( B_0 = 0 \) or \( B_0 = -A_0/k^2 \).
2. \( C_0 \) generic and \( B_j = -A_j \), \( B_0 = 0 \) or \( B_j = -A_j/k^2 \).

Under certain conditions \( Q_j(x, t) \) become periodic functions of \( x \), see [10, 11]. If the periods \( T_0, T_j \) satisfy

\[
\Theta_0(x + T_0) - \Theta_0(x) = 2\pi p_0, \quad \Theta_j(x + T_j) - \Theta_j(x) = 2\pi p_j \tag{18}
\]

for \( j = 1, \ldots, N_f \) then \( \Psi^0, \Psi^f_j \) will be periodic in \( x \) with periods \( T_0 = 2m_0\omega/\alpha \), \( T_j = 2m_j\omega/\alpha \). This holds true provided there exist pairs of integers \( m_0, p_0 \), and \( m_j, p_j \), such that

\[
m_0 = -\pi \left[ \alpha_0 \zeta(\omega) + \omega_0/\alpha \right]^{-1}, \quad \frac{m_j}{p_j} = -\pi \left[ \alpha_0 \zeta(\omega) + \omega_j/\alpha \right]^{-1}
\]

where \( \omega \) (and \( \omega' \)) are the half-periods of the Weierstrass functions \( \zeta \).

When \( V_{0,F} = V_{0,B} = V_0 \) and inserting (10) in (8) and equating the coefficients of equal powers of \( \sin(\alpha x, k) \) results in the following relations among the parameters \( \omega_j, C_j, A_j \) and \( B_j \) and the characteristic of the optical lattice \( V_0, \alpha \) and \( k \)

\[
\sum_{j=1}^{N_f} A_j = \frac{\alpha^2 k^2}{g_{BF}} \left( \frac{1}{m_B} - \frac{g_{BB}}{m_F g_{BF}} \right) - \frac{V_0}{g_{BF}} \left( 1 - \frac{g_{BB}}{g_{BF}} \right) \tag{19}
\]

\[
\omega_0 = \frac{\alpha^2 (k^2 + 1)}{2m_B} + g_{BB} B_0 + g_{BF} \sum_{i=1}^{N_f} B_i + \frac{\alpha^2 k^2}{2m_B} \frac{B_0}{A_0} \tag{20}
\]

\[
A_0 = \frac{\alpha^2 k^2 - m_F V_0}{m_F g_{BF}}, \quad \omega_j = \frac{\alpha^2 (k^2 + 1)}{2m_F} + g_{BB} B_0 + \frac{\alpha^2 k^2}{2m_F} \frac{B_j}{A_j}
\]

\[
C_0^2 = \frac{\alpha^2 B_0}{A_0} (A_0 + B_0)^2, \quad C_j^2 = \frac{\alpha^2 B_j}{A_j} (A_j + B_j)^2 \tag{21}
\]

where \( j = 1, \ldots, N_f \). Next for convenience we introduce

\[
B_0 = -\beta_0 A_0, \quad B_j = -\beta_j A_j, \quad j = 1, \ldots, N_f
\]

then

\[
C_0^2 = \alpha^2 A_0^2 \beta_0 (\beta_0 - 1)(1 - \beta_0 k^2), \quad C_j^2 = \alpha^2 A_j^2 \beta_j (\beta_j - 1)(1 - \beta_j k^2).
\]
Table 1. Constraints ensuring the existence of type A solutions. Here \( W = g_{BF}m_FW_B/(m_BW_F) \).

<table>
<thead>
<tr>
<th></th>
<th>( \beta_0 \leq 0 )</th>
<th>( \beta_j \leq 0 )</th>
<th>( A_0 \geq 0 )</th>
<th>( A_j \geq 0 )</th>
<th>( g_{BF} \gtrless 0 )</th>
<th>( g_{BB} \gtrless W )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>( A_0 \geq 0 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( \beta_0 \leq 0 )</td>
<td>( 1 \leq \beta_j \leq 1/k^2 )</td>
<td>( A_0 \geq 0 )</td>
<td>( A_j \leq 0 )</td>
<td>( g_{BF} \gtrless 0 )</td>
<td>( g_{BB} \gtrless W )</td>
</tr>
<tr>
<td>3</td>
<td>( 1 \leq \beta_0 \leq 1/k^2 )</td>
<td>( \beta_j \leq 0 )</td>
<td>( A_0 \leq 0 )</td>
<td>( A_j \geq 0 )</td>
<td>( g_{BF} \gtrless 0 )</td>
<td>( g_{BB} \gtrless W )</td>
</tr>
<tr>
<td>4</td>
<td>( 1 \leq \beta_0 \leq 1/k^2 )</td>
<td>( 1 \leq \beta_j \leq 1/k^2 )</td>
<td>( A_0 \leq 0 )</td>
<td>( A_j \leq 0 )</td>
<td>( g_{BF} \gtrless 0 )</td>
<td>( g_{BB} \gtrless W )</td>
</tr>
</tbody>
</table>

In order for our results (10) to be consistent with the parametrization (5)–(7) we must ensure that both \( q_0(x) \) and \( \Theta_0(x) \) are real-valued, and also \( q_j(x) \) and \( \Theta_j(x) \) are real-valued; this means that \( C_0^2 \geq 0 \) and \( q_0^2(x) \geq 0 \) and also \( C_j^2 \geq 0 \) and \( q_j^2(x) \geq 0 \) (see Table 1, \( W_B = (\alpha^2k^2 - m_BV_0) \), \( W_F = (\alpha^2k^2 - m_FV_0) \)). An elementary analysis shows that with \( l = 0, \ldots, N_f \) one of the following conditions must hold

a) \( A_l \geq 0, \quad \beta_l \leq 0 \)  
b) \( A_l \leq 0, \quad 1 \leq \beta_l \leq \frac{1}{k^2} \).

4. Type B Nontrivial Phase Solutions

For the first time solutions of this type were derived in [4,5] for the case of nonlinear Schrödinger equation and in [7] for the \( n \)-component CNLSE. For Bose–Fermi mixtures solutions of this type are possible

- when we have two lattices \( V_B \) and \( V_F \).
- when \( m_B = m_F \).

We seek the solutions in one of the following forms:

\[
\begin{align*}
q_0^2 &= A_0 \text{sn}(\alpha x, k) + B_0, & q_j^2 &= A_j \text{sn}(\alpha x, k) + B_j, & j = 1, \ldots, N_f \quad (22) \\
q_0^2 &= A_0 \text{cn}(\alpha x, k) + B_0, & q_j^2 &= A_j \text{cn}(\alpha x, k) + B_j \quad (23) \\
q_0^2 &= A_0 \text{dn}(\alpha x, k) + B_0, & q_j^2 &= A_j \text{dn}(\alpha x, k) + B_j \quad (24)
\end{align*}
\]

In the first case (22) we have

\[
\begin{align*}
V_B &= \frac{3\alpha^2k^2}{8m_B}, & V_F &= \frac{3\alpha^2k^2}{8m_F} \\
A_0 &= -\frac{\alpha^2k^2}{4m_F g_{BF}} B_j, & \frac{B_1}{A_1} &= \ldots = \frac{B_{N_f}}{A_{N_f}} \\
\sum_j A_j &= -\frac{\alpha^2k^2}{4m_B g_{BF}} B_0 - \frac{A_0 g_{BB}}{g_{BF}} \\
\omega_0 &= \frac{\alpha^2(k^2 + 1)}{8m_B} + g_{BB}B_0 + g_{BF}B_1 - \frac{\alpha^2k^2 B_0^2}{8m_B A_0^2}
\end{align*}
\]
\[ \omega_j = \frac{\alpha^2 (k^2 + 1)}{8m_F} + g_{BF} B_0 - \frac{\alpha^2 k^2 B_j^2}{8m_F A_j^2} \]

\[ C_0^2 = \frac{\alpha^2}{4A_0^2} (B_0^2 - A_0^2)(A_0^2 - B_0^2 k^2), \quad C_j^2 = \frac{\alpha^2}{4A_j^2} (B_j^2 - A_j^2)(A_j^2 - B_j^2 k^2). \]

We remark that due to relations \( B_{A_1} = \cdots = B_{A_N} \), we have that all \( q_j \) of the fermion fields are proportional to \( q_1 \).

### 5. Examples of Elliptic Solutions

Using the general solution equations (11)–(13) we have the following special cases (these solutions are possible only when we have some restrictions on \( g_{BB}, g_{BF}, \) and \( V_0 \), see Table 1):

**Example 1.** Suppose that \( B_0 = B_j = 0 \). Therefore we have

\[ q_0(x) = \sqrt{A_0} \text{sn}(\alpha x, k), \quad q_j = \sqrt{A_j} \text{sn}(\alpha x, k) \]

\[ A_0 = \frac{\alpha^2 k^2 - m_F V_0}{m_F g_{BF}}, \quad \sum_j A_j = \frac{\alpha^2 k^2}{g_{BF}} \left( \frac{1}{m_B} - \frac{g_{BB}}{m_F g_{FB}} \right) - \frac{V_0}{g_{BF}} \left( 1 - \frac{g_{BB}}{g_{BF}} \right). \]

For the frequencies \( \omega_0 \) and \( \omega_j \) we have

\[ \omega_0 = \frac{\alpha^2 (1 + k^2)}{2m_B}, \quad \omega_j = \frac{\alpha^2 (1 + k^2)}{2m_F}. \]

as well as \( C_0 = C_j = 0 \).

**Example 2.** Let \( B_0 = -A_0 \) and \( B_j = -A_j \) hold true. Then we have

\[ q_0(x) = \sqrt{-A_0} \text{cn}(\alpha x, k), \quad q_j(x) = \sqrt{-A_j} \text{cn}(\alpha x, k). \]

The coefficients \( A_0 \) and \( A_j \) have the same form as (26). The frequencies \( \omega_0 \) and \( \omega_j \) now look as follows

\[ \omega_0 = \frac{\alpha^2 (1 - 2k^2)}{2m_B} + V_0, \quad \omega_j = \frac{\alpha^2 (1 - 2k^2)}{2m_F} + V_0. \]

The constants \( C_0 \) and \( C_j \) are equal to zero again.

**Example 3.** \( B_0 = -A_0/k^2 \) and \( B_j = -A_j/k^2 \). In this case we obtain

\[ q_0(x) = \frac{\sqrt{-A_0}}{k} \text{dn}(\alpha x, k), \quad q_j(x) = \frac{\sqrt{-A_j}}{k} \text{dn}(\alpha x, k) \]

\[ \omega_0 = \frac{\alpha^2 (k^2 - 2)}{2m_B} + \frac{V_0}{k^2}, \quad \omega_j = \frac{\alpha^2 (k^2 - 2)}{2m_F} + \frac{V_0}{k^2}. \]
As before $C_0 = C_j = 0$.

**Example 4.** $B_0 = 0$ and $B_j = -A_j$. The result reads
\[
q_0(x) = \sqrt{A_0} \text{sn} (\alpha x, k), \quad q_j(x) = \sqrt{-A_j} \text{cn}(\alpha x, k)
\]
\[
\omega_0 = \frac{\alpha^2(1 - k^2)}{2 m_B} + V_0 + A_0 g_{BB}, \quad \omega_j = \frac{\alpha^2}{2 m_F}.
\]

By analogy with the previous examples the constants $A_0$, $A_j$, $C_0$ and $C_j$ are given by formulae (26) and $C_0$, $C_j$ are all zero.

**Example 5.** $B_0 = 0$ and $B_j = -\frac{A_j}{k^2}$. Thus, one gets
\[
q_0(x) = \sqrt{A_0} \text{sn} (\alpha x, k), \quad q_j(x) = \sqrt{-A_j} \text{cn}(\alpha x, k)
\]
\[
\omega_0 = \frac{\alpha^2(k^2 - 1)}{2 m_B} + \frac{V_0}{k^2} + \frac{A_0 g_{BB}}{k^2}, \quad \omega_j = \frac{\alpha^2 k^2}{2 m_F}.
\]

**Example 6.** Let $B_0 = -A_0$ and $B_j = 0$. Hence we have
\[
q_0(x) = \sqrt{-A_0} \text{cn} (\alpha x, k), \quad q_j(x) = \sqrt{A_j} \text{sn}(\alpha x, k)
\]
\[
\omega_0 = \frac{\alpha^2}{2 m_B} - g_{BB} A_0, \quad \omega_j = \frac{\alpha^2(1 - k^2)}{2 m_F} + V_0.
\]

**Example 7.** Let $B_0 = -A_0$ and $B_j = -\frac{A_j}{k^2}$. We obtain
\[
q_0(x) = \sqrt{-A_0} \text{cn} (\alpha x, k), \quad q_j(x) = \sqrt{A_j} \text{sn}(\alpha x, k)
\]
\[
\omega_0 = \frac{V_0}{k^2} - \frac{\alpha^2}{2 m_B} + \frac{1 - k^2}{k^2} A_0 g_{BB}, \quad \omega_j = V_0 - \frac{\alpha^2 k^2}{2 m_F}.
\]

**Example 8.** Suppose $B_0 = -\frac{A_0}{k^2}$ and $B_j = 0$. Then
\[
q_0(x) = \sqrt{-\frac{A_0}{k}} \text{dn} (\alpha x, k), \quad q_j(x) = \sqrt{A_j} \text{sn}(\alpha x, k)
\]
\[
\omega_0 = \frac{\alpha^2 k^2}{2 m_B} - \frac{g_{BB} A_0}{k^2}, \quad \omega_j = \frac{\alpha^2(k^2 - 1)}{2 m_F} + \frac{V_0}{k^2}.
\]

**Example 9.** Let $B_0 = -\frac{A_0}{k^2}$ and $B_j = -A_j$. Thus
\[
q_0(x) = \sqrt{-\frac{A_0}{k}} \text{dn} (\alpha x, k), \quad q_j(x) = \sqrt{-A_j} \text{cn}(\alpha x, k)
\]
\[
\omega_0 = V_0 - \frac{\alpha^2 k^2}{2 m_B} + \frac{k^2 - 1}{k^2} g_{BB} A_0, \quad \omega_j = \frac{V_0}{k^2} - \frac{\alpha^2}{2 m_F}.
\]

All these cases when $V_0 = 0$ and $j = 2$ are derived for the first time in [3].
Table 2. Constraints ensuring the existence of generic type B trivial phase solutions. Here $W = g_{BF} m_F W_B / (m_B W_F)$.

<table>
<thead>
<tr>
<th></th>
<th>$q_0 = \sqrt{A_0} \text{sn}(\alpha x, k)$</th>
<th>$q_j = \sqrt{A_j} \text{sn}(\alpha x, k)$</th>
<th>$g_{BF} \leq 0$</th>
<th>$g_{BB} \leq W$</th>
<th>$V_0 \leq \alpha^2 k^2 / m_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$q_0 = \sqrt{-A_0} \text{cn}(\alpha x, k)$</td>
<td>$q_j = \sqrt{-A_j} \text{cn}(\alpha x, k)$</td>
<td>$g_{BF} \leq 0$</td>
<td>$g_{BB} \leq W$</td>
<td>$V_0 \geq \alpha^2 k^2 / m_F$</td>
</tr>
<tr>
<td>2</td>
<td>$q_0 = \sqrt{-A_0} \text{dn}(\alpha x, k)/k$</td>
<td>$q_j = \sqrt{-A_j} \text{dn}(\alpha x, k)/k$</td>
<td>$g_{BF} \leq 0$</td>
<td>$g_{BB} \leq W$</td>
<td>$V_0 \geq \alpha^2 k^2 / m_F$</td>
</tr>
<tr>
<td>3</td>
<td>$q_0 = \sqrt{A_0} \text{sn}(\alpha x, k)$</td>
<td>$q_j = \sqrt{A_j} \text{sn}(\alpha x, k)$</td>
<td>$g_{BF} \geq 0$</td>
<td>$g_{BB} \geq W$</td>
<td>$V_0 \leq \alpha^2 k^2 / m_F$</td>
</tr>
<tr>
<td>4</td>
<td>$q_0 = \sqrt{A_0} \text{cn}(\alpha x, k)$</td>
<td>$q_j = \sqrt{-A_j} \text{dn}(\alpha x, k)/k$</td>
<td>$g_{BF} \geq 0$</td>
<td>$g_{BB} \geq W$</td>
<td>$V_0 \leq \alpha^2 k^2 / m_F$</td>
</tr>
<tr>
<td>5</td>
<td>$q_0 = \sqrt{A_0} \text{dn}(\alpha x, k)/k$</td>
<td>$q_j = \sqrt{-A_j} \text{sn}(\alpha x, k)$</td>
<td>$g_{BF} \geq 0$</td>
<td>$g_{BB} \geq W$</td>
<td>$V_0 \leq \alpha^2 k^2 / m_F$</td>
</tr>
<tr>
<td>6</td>
<td>$q_0 = \sqrt{-A_0} \text{cn}(\alpha x, k)$</td>
<td>$q_j = \sqrt{A_j} \text{sn}(\alpha x, k)$</td>
<td>$g_{BF} \geq 0$</td>
<td>$g_{BB} \geq W$</td>
<td>$V_0 \leq \alpha^2 k^2 / m_F$</td>
</tr>
<tr>
<td>7</td>
<td>$q_0 = \sqrt{-A_0} \text{cn}(\alpha x, k)$</td>
<td>$q_j = \sqrt{-A_j} \text{cn}(\alpha x, k)$</td>
<td>$g_{BF} \geq 0$</td>
<td>$g_{BB} \geq W$</td>
<td>$V_0 \leq \alpha^2 k^2 / m_F$</td>
</tr>
<tr>
<td>8</td>
<td>$q_0 = \sqrt{-A_0} \text{dn}(\alpha x, k)/k$</td>
<td>$q_j = \sqrt{A_j} \text{sn}(\alpha x, k)$</td>
<td>$g_{BF} \geq 0$</td>
<td>$g_{BB} \geq W$</td>
<td>$V_0 \leq \alpha^2 k^2 / m_F$</td>
</tr>
<tr>
<td>9</td>
<td>$q_0 = \sqrt{-A_0} \text{dn}(\alpha x, k)/k$</td>
<td>$q_j = \sqrt{-A_j} \text{sn}(\alpha x, k)$</td>
<td>$g_{BF} \geq 0$</td>
<td>$g_{BB} \geq W$</td>
<td>$V_0 \leq \alpha^2 k^2 / m_F$</td>
</tr>
</tbody>
</table>

5.1. Mixed Trivial Phase Solution

Example 10. When $B_0 = 0, B_1 = 0, B_2 = -A_2, B_j = -A_j/k^2, j = 3, \ldots, N_f$, the solutions obtain the form

$q_0 = \sqrt{A_0} \text{sn}(\alpha x, k), \quad q_1 = \sqrt{A_1} \text{sn}(\alpha x, k)$
$q_2 = \sqrt{-A_2} \text{cn}(\alpha x, k), \quad q_j = \sqrt{-A_j} \text{dn}(\alpha x, k)/k$.

Using equations (11)–(13) we have

$$A_0 = \frac{\alpha^2 k^2 - V_0 m_F}{m_F g_{BF}} \sum_{j=1}^{N_f} A_j = \alpha^2 k^2 \left( \frac{1}{m_B g_{BF}} - \frac{g_{BB}}{m_F g_{BF}^2} \right) - V_0 \left( \frac{1}{g_{BF}} - \frac{g_{BB}}{g_{BF}^2} \right)$$

$$\omega_0 = \frac{\alpha^2 (k^2 - 1)}{2m_B} + \frac{g_{BF}}{k^2} \left( A_1 + (1 - k^2) A_2 \right) + \frac{g_{BB} A_0}{k^2} + \frac{V_0}{k^2}$$
\( \omega_1 = \frac{\alpha^2(1 + k^2)}{2m_F}, \quad \omega_2 = \frac{1}{2m_F} \alpha^2, \quad \omega_j = \frac{\alpha^2 k^2}{2m_F}, \quad j = 3, \ldots, N_F. \)

**Example 11.** Let \( B_0 = B_1 = 0 \) and \( B_j = -A_j \) where \( j = 2, \ldots, N_f \). Therefore, the solutions read

\[
q_0(x) = \sqrt{-A_0} \, \text{sn}(\alpha x, k) \\
q_1(x) = \sqrt{-A_1} \, \text{sn}(\alpha x, k) \\
q_j(x) = \sqrt{-A_j} \, \text{cn}(\alpha x, k).
\]

Then we obtain for frequencies the following results

\[
\omega_0 = \frac{\alpha^2(1 - k^2)}{2m_B} + V_0 + g_{BB} A_0 + g_{BF} A_1, \quad \omega_1 = \frac{\alpha^2(1 + k^2)}{2m_F}, \quad \omega_j = \frac{\alpha^2 k^2}{2m_F}.
\]

**Example 12.** Suppose \( B_0 = -A_0, B_1 = 0, B_2 = -A_2 \) and \( B_j = -A_j/k^2 \) where \( j = 3, \ldots, N_f \). The solutions have the form

\[
q_0(x) = \sqrt{-A_0} \, \text{cn}(\alpha x, k), \quad q_1(x) = \sqrt{-A_1} \, \text{sn}(\alpha x, k) \\
q_2(x) = \sqrt{-A_2} \, \text{cn}(\alpha x, k), \quad q_j(x) = \sqrt{-A_j} \, \text{dn}(\alpha x, k)/k.
\]

The frequencies are

\[
\omega_0 = \frac{V_0}{k^2} - \frac{\alpha^2}{2m_B} + \frac{1 - k^2}{k^2} (g_{BB} A_0 + g_{BF} A_2) + \frac{g_{BF}}{k^2} A_1 \\
\omega_1 = V_0 + \frac{\alpha^2(1 - k^2)}{2m_F}, \quad \omega_2 = V_0 + \frac{\alpha^2(1 - 2k^2)}{2m_F}, \quad \omega_j = V_0 - \frac{\alpha^2 k^2}{2m_F}.
\]

**Example 13.** Let \( B_0 = -A_0, B_1 = -A_1 \) and \( B_j = -A_j/k^2 \) for \( j = 2, \ldots, N_f \). Then

\[
q_0(x) = \sqrt{-A_0} \, \text{cn}(\alpha x, k) \\
q_1(x) = \sqrt{-A_1} \, \text{cn}(\alpha x, k) \\
q_j(x) = \sqrt{-A_j} \, \text{dn}(\alpha x, k)/k
\]

\[
\omega_0 = \frac{V_0}{k^2} - \frac{\alpha^2}{2m_B} + \frac{1 - k^2}{k^2} (g_{BB} A_0 + g_{BF} A_1) \\
\omega_1 = V_0 + \frac{\alpha^2(1 - 2k^2)}{2m_F}, \quad \omega_j = V_0 - \frac{\alpha^2 k^2}{2m_F}.
\]
Example 14. Let $B_0 = -A_0/k^2$, $B_1 = -A_1$ and $B_j = -A_j/k^2$ for $j = 2, \ldots, N_f$. Hence

$$q_0(x) = \sqrt{-A_0} \frac{1}{\cosh(\alpha x)}$$
$$q_1(x) = \sqrt{-A_1} \frac{1}{\cosh(\alpha x)}$$
$$q_j(x) = \sqrt{-A_j} \frac{1}{\cosh(\alpha x)}$$

$$\omega_0 = \frac{\alpha^2(k^2 - 2)}{2m_B} + \frac{V_0}{k^2} + \frac{1 - k^2}{k^2} (g_{BB} A_0 + g_{BF} A_1)$$
$$\omega_1 = \frac{V_0}{k^2} - \frac{\alpha^2}{2m_F}, \quad \omega_j = \frac{V_0}{k^2} + \frac{\alpha^2(k^2 - 2)}{2m_F}.$$

Certainly these examples do not exhaust all possible combinations of solutions and it is easy to it.

6. Vector Soliton Solutions

6.1. Vector Bright-Bright Soliton Solutions

When $k \to 1$, $\text{sn}(\alpha x, 1) = \tanh(\alpha x)$ and $B_0 = -A_0$, $B_j = -A_j$ we obtain that the solutions read

$$q_0 = \sqrt{-A_0} \frac{1}{\cosh(\alpha x)}, \quad q_j = \sqrt{-A_j} \frac{1}{\cosh(\alpha x)}$$

where $A_0 \leq 0$ as well as $A_j \leq 0$. Using equations (11)–(13) we have

$$A_0 = \frac{\alpha^2 - V_0 m_F}{m_F g_{BF}}, \quad V = V_0 \tanh^2(\alpha x)$$
$$\sum_{j=1}^{N_f} A_j = \frac{\alpha^2}{g_{BF}} \left( \frac{1}{m_B} - \frac{g_{BB}}{m_F g_{BF}} \right) - \frac{V_0}{g_{BF}} \left( 1 - \frac{g_{BB}}{g_{BF}} \right)$$
$$\omega_0 = V_0 - \frac{1}{2m_B} \alpha^2, \quad \omega_j = V_0 - \frac{1}{2m_F} \alpha^2.$$

As a consequence of the restrictions on $A_0$ and $A_j$ one can get the following inequalities

$$g_{BF} > 0, \quad V_0 \geq \frac{\alpha^2}{m_F}, \quad g_{BB} \leq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_F V_0) m_B} g_{BF}$$
$$g_{BF} < 0, \quad V_0 \leq \frac{\alpha^2}{m_F}, \quad g_{BB} \geq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_F V_0) m_B} g_{BF}.$$

Vector bright soliton solution when $V_0 = 0$ is derived for the first time in [3].
6.2. Vector Dark-Dark Soliton Solutions

When \( k \to 1 \) and \( B_0 = B_j = 0 \) are satisfied the solutions read

\[
q_0(x) = \sqrt{A_0} \tanh(\alpha x), \quad q_j(x) = \sqrt{A_j} \tanh(\alpha x).
\]

The natural restrictions \( A_0 \geq 0 \) and \( A_j \geq 0 \) lead to

\[
g_{BF} > 0, \quad g_{BB} \leq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_F V_0) m_B} g_{BF}, \quad V_0 \leq \frac{\alpha^2}{m_F} \tag{31}
\]

\[
g_{BF} < 0, \quad g_{BB} \geq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_F V_0) m_B} g_{BF}, \quad V_0 \geq \frac{\alpha^2}{m_F}
\]

\[
A_0 = \frac{\alpha^2 - m_F V_0}{m_F g_{BF}}, \quad \sum_j A_j = \frac{\alpha^2}{g_{BF}} \left( \frac{1}{m_B} - \frac{g_{BB}}{m_B g_{BF}} \right) - \frac{V_0}{g_{BF}} \left( 1 - \frac{g_{BB}}{g_{BF}} \right).
\]

For the frequencies \( \omega_0 \) and \( \omega_j \) and the constants \( C_0 \) and \( C_j \) we have

\[
\omega_0 = \frac{\alpha^2}{m_B}, \quad \omega_j = \frac{\alpha^2}{m_F}, \quad C_0 = C_j = 0. \tag{32}
\]

6.3. Vector Bright-Dark Soliton Solutions

When \( k \to 1 \), \( B_0 = -A_0 \) and \( B_j = 0 \), we have

\[
q_0(x) = \frac{\sqrt{-A_0}}{\cosh(\alpha x)}, \quad q_j(x) = \sqrt{A_j} \tanh(\alpha x)
\]

\[
\omega_0 = \frac{\alpha^2}{2 m_B} - g_{BB} A_0, \quad \omega_j = V_0, \quad C_0 = C_j = 0.
\]

The parameters \( A_0 \) and \( A_j \) are given by (31). In this case we have the following restrictions

\[
g_{BF} > 0, \quad g_{BB} \geq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_F V_0) m_B} g_{BF}, \quad V_0 \geq \frac{\alpha^2}{m_F}
\]

\[
g_{BF} < 0, \quad g_{BB} \leq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_F V_0) m_B} g_{BF}, \quad V_0 \leq \frac{\alpha^2}{m_F}.
\]

6.4. Vector Dark-Bright Soliton Solutions

When \( k \to 1 \) and provided that \( B_0 = 0 \) and \( B_j = -A_j \) the result is

\[
q_0(x) = \sqrt{A_0} \tanh(\alpha x), \quad q_j(x) = \frac{\sqrt{-A_j}}{\cosh(\alpha x)}, \quad \omega_0 = V_0 + A_0 g_{BB}, \quad \omega_j = \frac{\alpha^2}{2 m_F}.
\]
By analogy with the previous examples the constants $A_0, A_j, C_0$ and $C_j$ are given by formulae (31) and (32), respectively. The restrictions now are

$$g_{BF} > 0, \quad g_{BB} \geq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_F V_0) m_B} g_{BF}, \quad V_0 \leq \alpha^2 / m_F$$

$$g_{BF} < 0, \quad g_{BB} \leq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_F V_0) m_B} g_{BF}, \quad V_0 \geq \alpha^2 / m_F.$$

### 6.5. Vector Dark-Dark-Bright Soliton Solutions

Let $B_0 = B_1 = 0$ and $B_j = -A_j$ where $j = 2, \ldots, N_f$. Therefore the solutions read

$$q_0(x) = \sqrt{A_0} \tanh(\alpha x), \quad q_1(x) = \sqrt{A_1} \tanh(\alpha x), \quad q_j(x) = \sqrt{-A_j} \text{sech}(\alpha x).$$

Then we obtain for frequencies the following results

$$\omega_0 = V_0 + g_{BB} A_0 + g_{BF} A_1, \quad \omega_1 = \frac{\alpha^2}{m_F}, \quad \omega_j = \frac{\alpha^2}{2m_F}.$$  

These examples are by no means exhaustive.

### 6.6. Nontrivial Phase, Trigonometric Limit

In this section we consider a trap potential of the form $V_{\text{trap}} = V_0 \cos(2\alpha x)$, as a model for an optical lattice. Our potential $V$ is similar and differs only with additive constant. When $k \to 0$, $\sin(\alpha x, 0) = \sin(\alpha x)$

$$q_0^2 = A_0 \sin^2(\alpha x) + B_0, \quad q_j^2 = A_j \sin^2(\alpha x) + B_j$$

$$V = V_0 \sin^2(\alpha x) = \frac{1}{2} (V_0 - V_0 \cos(2\alpha x)).$$

Using equations (11)–(13) again we obtain the following result when (see Table 3)

$$A_0 = -\frac{V_0}{g_{BF}} \sum_{j=1}^{N_f} A_j = -\frac{V_0}{g_{BF}} \left(1 - \frac{g_{BB}}{g_{BF}}\right)$$

<table>
<thead>
<tr>
<th></th>
<th>$\beta_0 \leq 0$</th>
<th>$\beta_j \leq 0$</th>
<th>$A_0 \geq 0$</th>
<th>$A_j \geq 0$</th>
<th>$g_{BF} \gg 0$</th>
<th>$g_{BB} \ll g_{BF}$</th>
<th>$V_0 \leq 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\beta_0 \leq 0$</td>
<td>$\beta_j \leq 0$</td>
<td>$A_0 \geq 0$</td>
<td>$A_j \geq 0$</td>
<td>$g_{BF} \gg 0$</td>
<td>$g_{BB} \ll g_{BF}$</td>
<td>$V_0 \leq 0$</td>
</tr>
<tr>
<td>2</td>
<td>$\beta_0 \leq 0$</td>
<td>$\beta_j \geq 1$</td>
<td>$A_0 \geq 0$</td>
<td>$A_j \leq 0$</td>
<td>$g_{BF} \gg 0$</td>
<td>$g_{BB} \gg g_{BF}$</td>
<td>$V_0 \leq 0$</td>
</tr>
<tr>
<td>3</td>
<td>$\beta_0 \geq 1$</td>
<td>$\beta_j \leq 0$</td>
<td>$A_0 \leq 0$</td>
<td>$A_j \geq 0$</td>
<td>$g_{BF} \gg 0$</td>
<td>$g_{BB} \ll g_{BF}$</td>
<td>$V_0 \geq 0$</td>
</tr>
<tr>
<td>4</td>
<td>$\beta_0 \geq 1$</td>
<td>$\beta_j \geq 1$</td>
<td>$A_0 \leq 0$</td>
<td>$A_j \leq 0$</td>
<td>$g_{BF} \gg 0$</td>
<td>$g_{BB} \ll g_{BF}$</td>
<td>$V_0 \geq 0$</td>
</tr>
</tbody>
</table>
\[\omega_0 = \frac{1}{2m_B} \alpha^2 + B_0 g_{BB} + g_{BF} \sum_{i=1}^{N_f} B_i, \quad \omega_j = \frac{1}{2m_F} \alpha^2 + g_{BF} B_0\]

\[\Omega_0^2 = \omega_0^2 B_0 (A_0 + B_0), \quad \Omega_j^2 = \omega_j^2 B_j (A_j + B_j)\]

where

\[\Theta_0(x) = \arctan \left( \sqrt{\frac{A_0 + B_0}{B_0}} \tan(\alpha x) \right)\]

\[\Theta_j(x) = \arctan \left( \sqrt{\frac{A_j + B_j}{B_j}} \tan(\alpha x) \right)\]

This solution is the most important from the physical point of view [16].

7. Linear Stability, Preliminary Results

To analyze linear stability of our initial system of equations we seek solutions in the form

\[\psi_0(x, t) = (q_0(x) + \epsilon \phi_0(x, t)) \exp \left( -\frac{i\omega_0}{\hbar} t + i\Theta_0(x) + i\kappa_0 \right)\]

\[\psi_j(x, t) = (q_1(x) + \epsilon \phi_j(x, t)) \exp \left( -\frac{i\omega_j}{\hbar} t + i\Theta_1(x) + i\kappa_1 \right)\]

and obtain the following linearized equations

\[
\begin{pmatrix}
\Phi_0 \\
\Phi_1 \\
\vdots \\
\Phi_{N_f}
\end{pmatrix}
= \begin{pmatrix}
\Lambda_0 & U_1 & U_2 & \ldots & U_{N_f} \\
V_1 & \Lambda_1 & 0 & \ldots & 0 \\
V_2 & 0 & \Lambda_2 & \ldots & 0 \\
& & & \ddots & \ddots \\
& & & & \Lambda_{N_f}
\end{pmatrix}
\begin{pmatrix}
\Phi_0 \\
\Phi_1 \\
\vdots \\
\Phi_{N_f}
\end{pmatrix}
\]

where

\[\Lambda_0 = \begin{pmatrix} S_0 & L_{0,-} \\ L_{0,+} & S_0 \end{pmatrix}, \quad U_j = \begin{pmatrix} 0 & 0 \\ U_{0,j} & 0 \end{pmatrix}\]

and

\[\Lambda_j = \begin{pmatrix} S_j & L_{j,-} \\ L_{j,+} & S_j \end{pmatrix}, \quad V_j = \begin{pmatrix} 0 & 0 \\ U_{1,j} & 0 \end{pmatrix}\]

\[S_0 = -\frac{C_0}{m_B q_0} \partial_x \left( \frac{1}{q_0} \right), \quad S_j = -\frac{C_j}{m_F q_j} \partial_x \left( \frac{1}{q_j} \right)\]
The analysis of the latter matrix system is a difficult problem and only numerical simulations are possible. Recently a great progress was achieved for analysis of linear stability of periodic solutions of type (5), (6) (see, e.g., [4, 5, 7, 11] and references therein). Nevertheless the stability analysis is known only for solutions of type (25)–(30) and solutions with nontrivial phase of type (33) and (34). Linear analysis of soliton solutions is well developed, but it is out scope of the present paper.

Finally we discuss three special cases:

**Case I.** Let \( B_0 = B_j = 0 \) then for \( j = 1, \ldots, N_f \) and \( q_0 = \sqrt{A_0} \text{sn}(\alpha x, k), \ q_j = \sqrt{A_j} \text{sn}(\alpha x, k) \) we have the following linearized equations

\[
\begin{align*}
\hbar \phi_{0,t}^R &= - \frac{1}{2m_B} \partial_{xx}^2 \phi_0^R + \left( V_0 + g_{BB} A_0 + g_{BF} \sum_j A_j \right) \text{sn}^2(\alpha x, k) \phi_0^I - \omega_0 \phi_0^I \\
\hbar \phi_{0,t}^I &= \frac{1}{2m_B} \partial_{xx}^2 \phi_0^I - \left( V_0 + 3g_{BB} A_0 + g_{BF} \sum_j A_j \right) \text{sn}^2(\alpha x, k) \phi_0^R + \omega_0 \phi_0^R - 2g_{BF} A_0 \text{sn}^2(\alpha x, k) \sum_j \phi_j^R \\
\hbar \phi_{j,t}^R &= - \frac{1}{2m_F} \partial_{xx}^2 \phi_j^R + (V_0 + g_{BF} A_0) \text{sn}^2(\alpha x, k) \phi_j^I - \omega_j \phi_j^I \\
\hbar \phi_{j,t}^I &= \frac{1}{2m_F} \partial_{xx}^2 \phi_j^I - (V_0 + g_{BF} A_0) \text{sn}^2(\alpha x, k) \phi_j^R + \omega_j \phi_j^R - 2g_{BF} \sqrt{A_0 A_j} \text{sn}^2(\alpha x, k) \phi_0^R.
\end{align*}
\]
Case II. Let $B_0 = -A_0$, $B_j = -A_j$ then for $q_0 = \sqrt{-A_0} \text{cn}(\alpha x, k)$, $q_j = \sqrt{-A_j} \text{cn}(\alpha x, k)$ we obtain the following linearized equations

$$
\hbar \phi_{0,t} = -\frac{1}{2m_B} \partial_{xx}^2 \phi_0^I + \left( V_0 + g_{BB} A_0 + g_{BF} \sum_j A_j \right) \text{sn}^2(\alpha x, k) \phi_0^I
$$

$$
\hbar \phi_{0,t} = -\left( g_{BB} A_0 + g_{BF} \sum_j A_j + \omega_0 \right) \phi_0^I
$$

$$
\hbar \phi_{0,t} = \frac{1}{2m_B} \partial_{xx}^2 \phi_0^R + \left( 3g_{BB} A_0 + g_{BF} \sum_j A_j + \omega_0 \right) \phi_0^R
$$

$$
- \left( V_0 + 3g_{BB} A_0 + g_{BF} \sum_j A_j \right) \text{sn}^2(\alpha x, k) \phi_0^R
$$

$$
- 2g_{BF} A_0 (1 - \text{sn}^2(\alpha x, k)) \sum_j \phi_j^R
$$

$$
\hbar \phi_{j,t} = -\frac{1}{2m_B} \partial_{xx}^2 \phi_j^I + \left( V_0 + g_{BF} A_0 \right) \text{sn}^2(\alpha x, k) \phi_j^I - \left( g_{BF} A_0 + \omega_j \right) \phi_j^I
$$

$$
\hbar \phi_{j,t} = \frac{1}{2m_B} \partial_{xx}^2 \phi_j^R - \left( V_0 + g_{BF} A_0 \right) \text{sn}^2(\alpha x, k) \phi_j^R + \left( g_{BF} A_0 + \omega_j \right) \phi_j^R
$$

$$
- 2g_{BF} \sqrt{A_0 A_j (1 - \text{sn}^2(\alpha x, k))} \phi_0^R , \quad j = 1, \ldots, N_f.
$$

Case III. Let $B_0 = -A_0/k^2$, $B_j = -A_j/k^2$ therefore the solutions are

$$
q_0 = \sqrt{-A_0} \text{dn}(\alpha x, k)/k, \quad q_j = \sqrt{-A_j} \text{dn}(\alpha x, k)/k
$$

and we obtain the following linearized equations

$$
\hbar \phi_{0,t} = -\frac{1}{2m_B} \partial_{xx}^2 \phi_0^I + \left( V_0 + g_{BB} A_0 + g_{BF} \sum_j A_j \right) \text{sn}^2(\alpha x, k) \phi_0^I
$$

$$
- \left( g_{BB} A_0 + g_{BF} \sum_j A_j + k^2 \omega_0 \right) \phi_0^I / k^2
$$

$$
\hbar \phi_{0,t} = \frac{1}{2m_B} \partial_{xx}^2 \phi_0^R + \left( 3g_{BB} A_0 + g_{BF} \sum_j A_j + k^2 \omega_0 \right) \phi_0^R / k^2
$$

$$
- \left( V_0 + 3g_{BB} A_0 + g_{BF} \sum_j A_j \right) \text{sn}^2(\alpha x, k) \phi_0^R
$$

$$
- 2g_{BF} A_0 (1 - \text{sn}^2(\alpha x, k)) \sum_j \phi_j^R
$$

$$
\hbar \phi_{j,t} = -\frac{1}{2m_B} \partial_{xx}^2 \phi_j^I + \left( V_0 + g_{BF} A_0 \right) \text{sn}^2(\alpha x, k) \phi_j^I - \left( g_{BF} A_0 + \omega_j \right) \phi_j^I
$$

$$
\hbar \phi_{j,t} = \frac{1}{2m_B} \partial_{xx}^2 \phi_j^R - \left( V_0 + g_{BF} A_0 \right) \text{sn}^2(\alpha x, k) \phi_j^R + \left( g_{BF} A_0 + \omega_j \right) \phi_j^R
$$

$$
- 2g_{BF} \sqrt{A_0 A_j (1 - \text{sn}^2(\alpha x, k))} \phi_0^R , \quad j = 1, \ldots, N_f.
$$
+ \frac{2g_{BF}A_0(1 - k^2 \text{sn}^2(\alpha, k))}{k^2} \sum_j \phi_j^R

\hbar \phi_{j,t}^R = - \frac{1}{2m_F} \frac{\partial^2 R}{\partial x^2} \phi_j^R + (V_0 + g_{BF}A_0) \text{sn}^2(\alpha x, k) \phi_j^I

- \frac{g_{BF}A_0 + k^2 \omega_j}{k^2} \phi_j^I

\hbar \phi_{j,t}^I = \frac{1}{2m_F} \frac{\partial^2 I}{\partial x^2} \phi_j^I - (V_0 + g_{BF}A_0) \text{sn}^2(\alpha x, k) \phi_j^R

+ \frac{g_{BF}A_0 + k^2 \omega_j}{k^2} \phi_j^R

- \frac{2g_{BF} \sqrt{A_0} \text{sn}^2(\alpha, k))}{k^2} \phi_0^R, \quad j = 1, \ldots, N_f.

These cases are by no means exhaustive.

8. Conclusions

In conclusion, we have considered the mean field model for boson-fermion mixtures in two optical lattices. Classes of quasi-periodic, periodic, elliptic solutions have been analyzed. These solutions can be used as initial states which can generate localized matter waves (solitons) through the modulational instability mechanism. This important problem is under consideration.

Acknowledgements

We thank the Bulgarian Science Foundation for a partial support under the contract # F-1410.

References


